4.6 - On the exact exponent for LIS:

Some natural questions are:

① Is there an exponent for the LIS of μ_p^S-random permutations? That is, is there $d(p)>0$ s.t.

$$\mathbb{P}\left(LIS(\sigma_n) = n^{d(p)+o(1)} \right) \xrightarrow[n\to\infty]{} 1 \ ?$$

② If the answer to the previous question is YES, what is the exact value of $d(p)$? For the moment we know that $\alpha_*(p) \leq d(p) \leq \beta^*(p)$.

The first thing one can do for question ② are simulations:

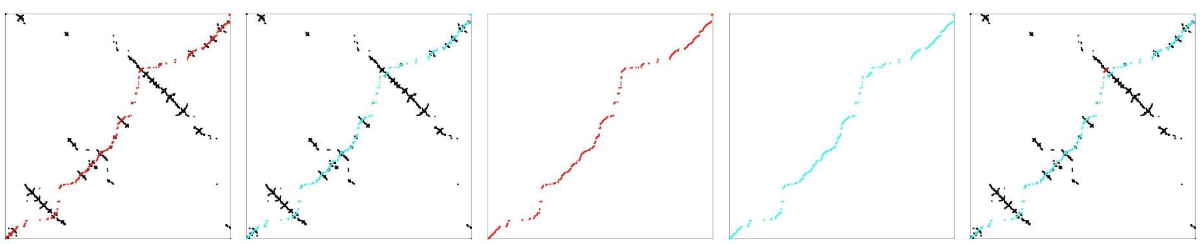


Figure: **From left to right:** (1) A permutation of length 262144 sampled from the Brownian separable permuton $\mu^{S}_{1/2}$ with one longest increasing subsequence in red of length 22546. (2) The same permutation with one increasing subsequence of length 21751 in cyan computed using our selection rule S in (1.6). (3-4) The two diagrams in (1) and (2) with only the two increasing subsequences. (5) The cyan increasing subsequence is plotted on top of the red increasing subsequence. Note that the two sequences are very similar since the cyan subsequence almost completely covers the red subsequence.

$\mathrm{LIS}(\mathrm{Perm}(\boldsymbol{\mu}_p, n)) \approx C(p) \cdot n^{d(p)}$			
p	$\alpha_*(p)$	$d(p)$	$\lvert d(p) - \alpha_*(p)\rvert$
0.1	0.584	0.58 ± 10^{-2}	$< 2 \cdot 10^{-2}$
0.2	0.653	$0.656 \pm 7 \cdot 10^{-3}$	$< 1.4 \cdot 10^{-2}$
0.3	0.712	$0.715 \pm 5 \cdot 10^{-3}$	$< 10^{-2}$
0.4	0.765	$0.766 \pm 4 \cdot 10^{-3}$	$< 8 \cdot 10^{-3}$
0.5	0.812	$0.814 \pm 3 \cdot 10^{-3}$	$< 6 \cdot 10^{-3}$
0.6	0.855	$0.857 \pm 2 \cdot 10^{-3}$	$< 4 \cdot 10^{-3}$
0.7	0.895	$0.897 \pm 2 \cdot 10^{-3}$	$< 4 \cdot 10^{-3}$
0.8	0.932	0.933 ± 10^{-3}	$< 2 \cdot 10^{-3}$
0.9	0.967	0.967 ± 10^{-3}	$< 2 \cdot 10^{-3}$

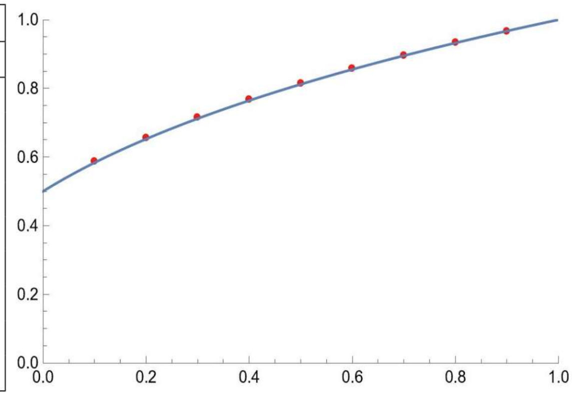


Figure: **Left:** For various values of the parameter p (first column) we indicate the value of our lower bound $\alpha_*(p)$ (second column), the value of the exponent $d(p)$ estimated from simulations (third column), and the difference between these two values (fourth column). **Right:** In blue, the plot of the function $\alpha_*(p)$. In red, the numerical values for the exponents $d(p)$ (computed numerically).

MORAL: The lower bound $\alpha_*(p)$ should be very close to the exact value $d(p)$

Q: Are they the same?

[Unpublished]

Together with A. Adhikari & W. Da Silva we proved that $\alpha_*(p) < d(p)$ $\forall p \in (0,1)$.

Coming back to question ①, we have the following result:

Theorem [Adhikari, B., Budzinski, Da Silva, Sénizergues, 24⁺]

There exists $d(p) > 0$ s.t.

$$\mathbb{P}\left(\mathrm{LIS}(\sigma_n) = n^{d(p)+o(1)} \right) \xrightarrow[n\to\infty]{} 1.$$

To prove the theorem, we first need a definition.

Def: Let $n \geq 0$, $K \geq 1$ be integers. Let $\alpha_1, \dots, \alpha_K > 0$ and $\alpha_0 = \sum_{i=1}^{K} \alpha_i$.
The discrete-multinomial distribution $\mathrm{Dir}(n; \alpha_1, \dots, \alpha_K)$ is the law on K-tuples of non-negative integers summing to n given by

$$\mathrm{Dir}(n; \alpha_1, \dots, \alpha_K)(n_1, \dots, n_K) = \frac{\Gamma(\alpha_0)\Gamma(n+1)}{\Gamma(n+\alpha_0)} \prod_{i=1}^{K} \frac{\Gamma(n_i + \alpha_i)}{\Gamma(\alpha_i)\Gamma(n_i+1)}.$$

In oder words, if $N = (N_1, \dots, N_K) \sim \mathrm{Dir}(n; \alpha_1, \dots, \alpha_K)$, then if

$$W = (W_1, \dots, W_K) \sim \mathrm{Dir}(\alpha_1, \dots, \alpha_K)$$

$\left[f_{\alpha_1, \dots, \alpha_K}(w_1, \dots, w_K) = \frac{1}{C} \prod_{i=1}^{K} w_i^{\alpha_i - 1} \right]$ with $\sum_{i=1}^{K} x_i = 1$.

then, conditionally on W,

$$N \sim \mathrm{Multinomial}(W).$$

$\left[f_{w_1, \dots, w_K}(n_1, \dots, n_K) = \frac{n!}{x_1! \cdots x_K!} w_1^{n_1} \cdots v_K^{n_K} \right]$ with $\sum_{i=1}^{K} n_i = n$

Let $X_n = \mathrm{LIS}(\sigma_n)$

Lemma: For all $n \geq 1, m \geq 1$, we have the stochastic domination

$$X_n \succcurlyeq \sum_{i=1}^{X_m} X^{(i)}_{N_i}$$

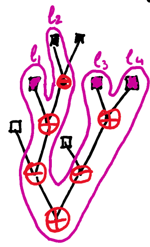
[That is, $\exists$ a coupling s.t. $X \geq Y$ a.s.]

where:

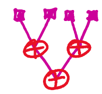
- $(\tilde{N}_i)_{1 \leq i \leq 2m-1} \sim \mathrm{Dir}\left(n-m; \frac{1}{2}, \dots, \frac{1}{2}\right)$ with $\begin{cases} \tilde{N}_i = N_i - 1 & \text{if } 1 \leq i \leq m \\ \tilde{N}_i = N_i & \text{if } m+1 \leq i \leq 2m+1 \end{cases}$
- Conditionally on $(N_i)_i$, $X^{(i)}_{N_i}$ are iid with law X_{N_i}.

To prove this theorem, it is useful to take the following point-of-view:

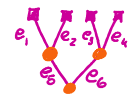
$\mathrm{LIS}(\sigma_n) \overset{d}{=}$ Largest subtree spanned by leaves and containing only $\oplus$ at the branching points of a uniform binary tree with n leaves and iid p-signs on its internal vertices.

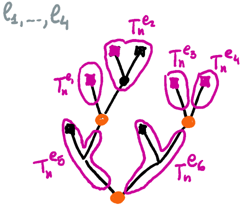


T_n

$T_n|_{\ell_1,\ell_2,\ell_3,\ell_4} =$ 

We also have the following correspondence between edges of $T_n|_{\ell_1,\dots,\ell_4}$ and subtrees of T_n





Proof of the lemma:

- Let T_n be a uniform binary tree with n leaves and iid p-signs on internal vertices.
- Let $S_n^m = \{\ell_1, \dots, \ell_m\}$ be m uniform leaves of T_n.
- $T_n|_{S_n^m}$ is a binary tree with m leaves and $m-1$ internal nodes.
- Label the edges of $T_n|_{S_n^m}$ by $e_1, \dots, e_{2m-1}$ as shown above
- Let $N_i = \#$ of leaves of T_n in $T_n^{e_i}$

Facts:

- Conditionally on $T_n|_{S_n^m}$, the vector $(\tilde{N}_i)_i \sim \mathrm{Dir}(n-m; \frac{1}{2}, \dots, \frac{1}{2})$
- Conditionally on $T_n|_{S_n^m}$ and $(\tilde{N}_i)_i$, $T_n^{e_i} \overset{d}{=} T_{N_i}$.

Choose one if there are multiple ones.

- Now consider the largest induced subtree of $T_n|_{S_n^m}$ with only $\oplus$-signs, and denote by $L_n^m \subseteq \{\ell_1, \dots, \ell_m\}$ the set of its leaves.
- Now $\# L_n^m \overset{d}{=} X_m$.
- Moreover, for each $\ell_i \in L_n^m$ consider a maximal subtree $\hat{T}_n^{e_i}$ of $T_n^{e_i}$ with only $\oplus$-signs.
- The subtree of T_n induced by leaves of $\bigcup_{\ell_i \in L_n^m} \hat{T}_n^{e_i}$ has only $\oplus$-signs by construction
- $\#$ of leaves of such tree $\overset{d}{=} \sum_{i=1}^{X_m} X_{N_i}^{(i)}$. $\square$

We can now complete the proof of the theorem.

F. M. Dekking and G. R. Grimmett. Superbranching processes and projections of random cantor sets. Probability Theory and Related Fields, 78:335–355, 1988

Indeed in this case we can apply the (simpler) arguments in the article above.

Proof of the Theorem:

We know that

$$X_n \geq \sum_{i=1}^{X_m} X^{(i)}_{N_i}$$

(We would love to have something like $Y_n \geq \sum_{i=1}^{Y_m} Y^{(i)}_{n/m}$)

Intuitively $N_i \approx n \cdot B_i$ where $(B_1, \dots, B_{2m-1}) \sim \mathrm{Dir}(\frac{1}{2}, \dots, \frac{1}{2}) \Rightarrow N_i \approx C_i \cdot \frac{n}{m}$

C_i: some random variable

To do that, we fix $a>0$ and consider only the i's s.t. $N_i > a\frac{n}{m}$:

$$\circledast \quad X_n \geq \sum_{i=1}^{X_m} X^{(i)}_{N_i} \geq \sum_{i=1}^{X_m} X^{(i)}_{N_i} \mathbb{1}_{\{N_i > a \cdot n/m\}} \geq \sum_{i=1}^{Y^n_m} X^{(i)}_{\frac{a}{m}\cdot n} \quad \text{where } Y^n_m = \sum_{i=1}^{X_m} \mathbb{1}_{\{N_i > a \cdot \frac{n}{m}\}}$$

Note that

$$\mathbb{E}[Y^n_m] = \mathbb{E}[X_m] \cdot \mathbb{P}\left(N_i > a \frac{n}{m}\right)$$

$$\downarrow n \to \infty$$

$$\mathbb{P}\left(\mathrm{Beta}(1/2, (m-1)) \geq \frac{a}{m}\right) \geq b$$

for some fixed $b>0$ and uniformly in m.

We now introduce some notation:

- Take n_0 large enough so that $\mathbb{E}[Y^n_m] \geq \mathbb{E}[X_m] \cdot b \quad \forall n \geq n_0 \quad \circledast$
- Let $n \geq n_0$ and take k to be the unique integer such that

$$\triangle \quad \left(\frac{m}{a}\right)^k \leq \frac{n}{n_0} \leq \left(\frac{m}{a}\right)^{k+1}$$

- Let $m \geq 1$ and write $d_m := \dfrac{\log \mathbb{E}[X_m]}{\log(m)}$ (so that $\mathbb{E}[X_m] = m^{d_m}$)

We now argue that

$$X_n \geq \text{Size of } K\text{-th generation of } \underline{\text{branching process}} \text{ started with } 1 \text{ particle and with reproduction law } Y_m^n$$

$(B_i)_{i\geq 1}$

Claim: $X_n \geq \sum_{i=1}^{B_j} X^{(i)}_{(\frac{a}{m})^j \cdot n} \quad \forall \; 0 \leq j \leq K.$

Proof: By induction. If $j=0$ we get $X_n \geq X_n$. The inductive step is an application of ⊛ with $(\frac{a}{m})^j \cdot n$ instead of n. □

By ⊛, the offspring distribution of $(B_i)_{i\geq 1}$ is so that:

$$\mathbb{E}\left[Y_m^n\right] \geq b \cdot m^{d_m} \quad , \quad \forall\, n \geq n_0.$$

and so, by the Claim above $\mathbb{E}\left[X_{(\frac{m}{a})^K}\right] \geq \mathbb{E}\left[\sum_{i=1}^{B_K} X_1^{(i)}\right] = \mathbb{E}\left[B_K\right] = \mathbb{E}\left[Y_m^n\right]^K \geq \left(b \cdot m^{d_m}\right)^K.$

Hence: $$\mathbb{P}\left(X_{(\frac{m}{a})^K} \geq c \cdot \left(b \cdot m^{d_m}\right)^K\right) \geq c \;, \quad \text{for all } K.$$

If this event occour, then:

by Δ

$n \geq (\frac{m}{a})^K \cdot n_0 \geq (\frac{m}{a})^K$ by Δ $\qquad (\frac{m}{a})^{K+1} \geq \frac{n}{n_0} \Rightarrow K \geq \frac{\log(n/n_0)}{\log(m/a)} - 1$

$$X_n \geq c \cdot \exp\left(K\left(\log b + d_m \log m\right)\right) \geq c \cdot \exp\left(\left(\frac{\log(n/n_0)}{\log(m/a)} - 1\right)\left(\log(b) + d_m \log(m)\right)\right)$$

and so

$$\frac{\log X_n}{\log(n)} \geq d_m \frac{\log(m)}{\log(m/a)} + \frac{\log(b)}{\log(m/a)} + O\left(\frac{1}{\log(n)}\right)$$

May depend on m.

We now set $d = \lim_m p\, d_m$.

By Markov's inequality, $\forall \varepsilon > 0$

$$\mathbb{P}\left(X_n > {}^{d+\varepsilon} \right) \xrightarrow[n\to\infty]{} 0 \qquad \left(\text{Recall that } \mathbb{E}[X_n] = n^{d_n}\right)$$

For the other inequality: Fix $\varepsilon > 0$ and choose m large enough so that

$$\frac{\log(m)}{\log(m/a)} \geq 1-\varepsilon, \qquad \frac{\log(b)}{\log(m/a)} \leq \varepsilon, \qquad d_m \geq d-\varepsilon.$$

Then $*$ implies that $\exists\, c > 0$ s.t.

$$** \quad \mathbb{P}\left(\frac{\log(X_n)}{\log(n)} \geq d - 4\varepsilon \right) \geq c, \quad \text{for } n \text{ large enough.}$$

Finally we need to increase this probability from c to $1-\varepsilon$. [Using independence]

Fix $m' > 0$ large enough, so that

$$* \quad \mathbb{P}\left(Y_{m'}^{n} > \frac{1}{\varepsilon \cdot c} \right) \geq 1-\varepsilon$$

Then we have that

$$\mathbb{P}\left(X_n \leq \left(\frac{a}{m'} \cdot n\right)^{d-4\varepsilon} \right) \overset{\circledast}{\leq} \mathbb{P}\left(Y_{m'} \leq \frac{1}{\varepsilon c} \right) + \mathbb{P}\left(\forall\ 1 \leq i \leq \frac{1}{\varepsilon \cdot c},\ X^{(i)}_{\frac{a}{m'} \cdot n} \leq \left(\frac{a}{m'} \cdot n\right)^{d-4\varepsilon} \right)$$

$$\overset{(*+**)}{\leq} \varepsilon + (1-c)^{\frac{1}{\varepsilon c}}$$

$$\leq \varepsilon + \exp(-\varepsilon^{-1})$$

$$\leq 2\varepsilon.$$

Hence, for n large enough, with probability at least $1-2\varepsilon$, we have

$$X_n \geq \left(\frac{a}{m'} \cdot n\right)^{d-4\varepsilon} \geq n^{d-5\varepsilon} \qquad \square$$

4.7 – On the exact value of $d(p)$

Recently, we had a new idea that lead us to the following conjecture:

Conjecture [Adhikari, B., Budzinski, Da Silva, Sénizergues, 24+]

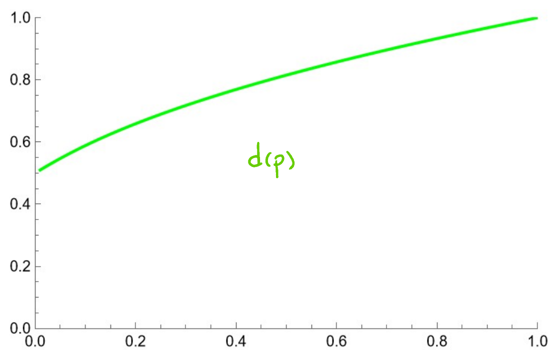
For $p \in (0,1)$, the value $d(p)$ is determined by

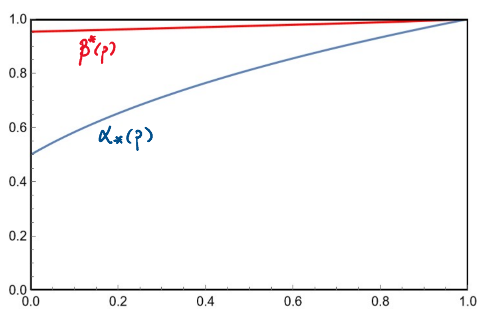
$$d(p) = \frac{1}{2(\gamma(p)-1)}$$

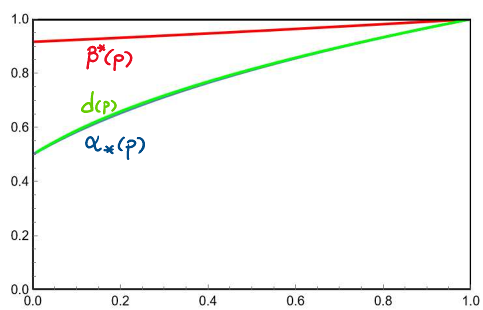
where $\gamma(p)$ solves the following equation in γ:

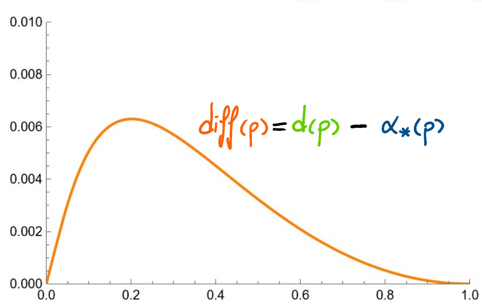
$$\frac{1-p+p\cdot 2^{\gamma-1}}{\gamma-1} = p\int_0^{1/2} x^{-\gamma}\left((1-x)^{-\gamma}-1\right)dx$$

Some graphics:









Note that the difference between our lower bound $\alpha_*(p)$ and $d(p)$ is at most ≈ 0.006, but the lower bound is always off !

Remark: We have a full strategy to prove the conjecture!

4.8 - Future reseach directions

Recall that μ^S_ρ is a specific instance of the SKEW BROWNIAN PERMUTON $\mu_{\rho,q}$.

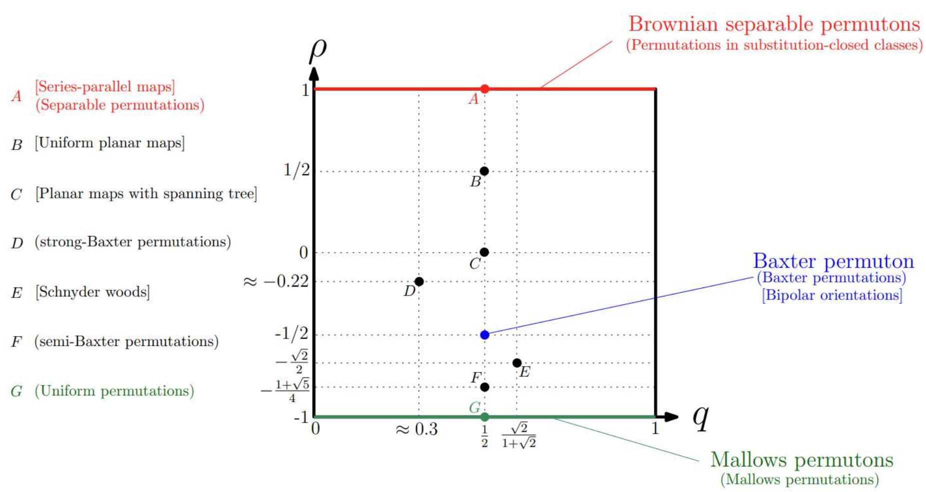
We have the following conjecture:

Conjecture [B., Da Silva, Gwynne, 23]

Let σ_n be sampled from $\mu_{\rho,q}$ for some $\rho \in (-1,1]$, $q \in [0,1]$. Then

$$\mathbb{P}\left(LIS(\sigma_n) = n^{d(\rho,q)+o(1)} \right) \xrightarrow[n\to\infty]{} 1$$

where $d(-1,q) = \frac{1}{2}$, $\forall q \in [0,1]$, $d(1,q) = d(1-q)$, $d(-\frac{1}{2},\frac{1}{2}) = 3/4$, and $d(\rho,q)$ is continuous, non-increasing in q and non-decreasing in ρ.



Moreover, the bijections between

series-parallel maps $\longleftrightarrow$ seperable permutations

Bipolar orientations $\longleftrightarrow$ Baxter permutations

suggest that there should be a natural DIRECTED METRIC ON PLANAR MAPS where the length of a geodesic is of ordere

$$n^{d(\rho,q)+o(1)}$$

and this metric should have a certain scaling limit which should be a good notion of DIRECTED-LQG-METRIC which will depend on ρ and q.

Finally since $\mu_{\rho,\frac{1}{2}}$ is conjectured to converge to the Leb. measure when $\rho \to 1$, we believe that if

$$\sigma_n^{\rho,\frac{1}{2}} \text{ is sampled from } \mu_{\rho,\frac{1}{2}}$$

then $LIS(\sigma_n^{\rho,\frac{1}{2}})$ behaves like LIS for a uniform permutation, when $\rho \to 1$.

This also suggests that it is possible that the directed-LQG-metric above is connected with the directed landscape when $\rho \to 1$.

Hopefully, one day we will know what is the TRUTH. :)