Permutations in Random Geometry

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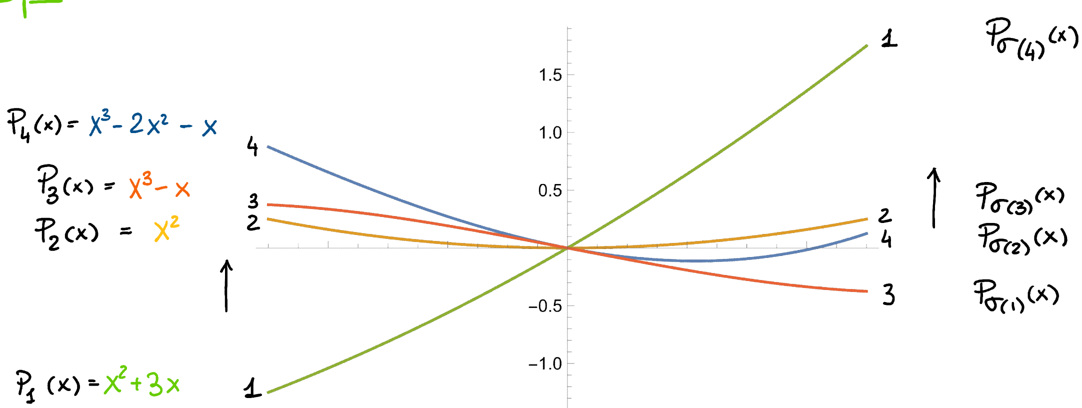
1 - Introduction: two canonical models of constrained permutations

1.1 - Polynomial interchanges & separable permutations (and friends)

Consider 4 polynomials $P_1(x), P_2(x), P_3(x), P_4(x)$ s.t. $P_i(0) = 0$ $\forall i = 1,2,3,4$.
Comparing the values of $P_i(x)$ for small $x<0$ and small $x>0$ yields a permutation σ.

Example:



$\sigma = 3\,4\,2\,1$ $\rightsquigarrow$ [One line notation for permutations: $\sigma = \sigma(1)\,\sigma(2) \ldots \sigma(n)$]

Observation: [Kontsevich, 2009]

For any choice of four polynomials $(P_i(x))_{i=1}^{4}$, then the corresponding $\sigma \in S_4 \setminus \{2413, 3142\}$

Def: A permutations $\sigma \in S_n$ is a POLYNOMIAL INTERCHANGE if $\exists$ n polynomials $P_1(x), \ldots, P_n(x)$ with $P_i(0) = 0$ $\forall i$ and such that:

- $P_1(x) < P_2(x) < \ldots < P_n(x)$ for small $x<0$;
- $P_{\sigma(1)}(x) < P_{\sigma(2)}(x) < \ldots < P_{\sigma(n)}(x)$ for small $x>0$.

Def: A permutation $\sigma \in S_n$ is **SEPARABLE** if it avoids the patterns 2413 and 3142, that is, if $\nexists$ four indeces $1 \le i_1 < \ldots < i_4 \le n$ such that

$$\sigma(i_2) < \sigma(i_4) < \sigma(i_1) < \sigma(i_3) \quad \text{or} \quad \sigma(i_3) < \sigma(i_1) < \sigma(i_4) < \sigma(i_2).$$

See Chapter 1 of this book for a proof.

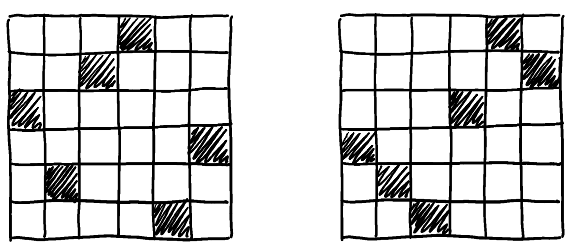
THEOREM [Étienne Ghys, "A singular mathematical promenade"]

A permutation is a polynomial interchange iff it is separable.

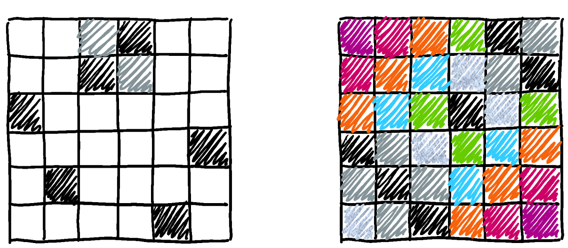
The proof of this theorem is NOT too hard but it uses certain structure properties of separable permutations that we have no time to see here.

Another interesting characterization of separable permutations is as follows:

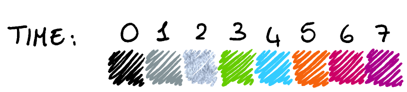
Consider Bootstrap percolation on a $n \times n$ grid with one sick square per each line and row (as initial configuration), i.e. with a permutation as initial configuration:



Question: Is everyone infected in the end? (At every step of the process, each box with two sick neighbours gets infected)



NO! YES!



THEOREM: [Shapiro, Stephens, 1991]

Consider bootstrap percolation on the $n \times n$ grid with a permutation as initial configuration.

"Everyone is infected in the end" $\iff$ the initial permutation configuration is SEPARABLE

Separable permutations pops up "here and there" in the literature in many different contexts, for instance, they are known to be in bijection with a model of planar maps called series-parallel maps and a family of trees called Schröder trees.

no loop-edges and no cut-vertices (i.e. vertices that disconnect the map if removed)

Def: A rooted non-separable planar map is SERIES-PARALLEL if it does not contain K_4 as minor.

Def: A SCHRÖDER TREE is a planar rooted tree with all the internal vertices of degree at least two and decorated by a plus or a minus. Moreover, plus and minus decorations do alternate along the branches.

THEOREM: [Bonichon, Bousquet-Mélou, Fusy, 2008]

The following objects are in bijection:

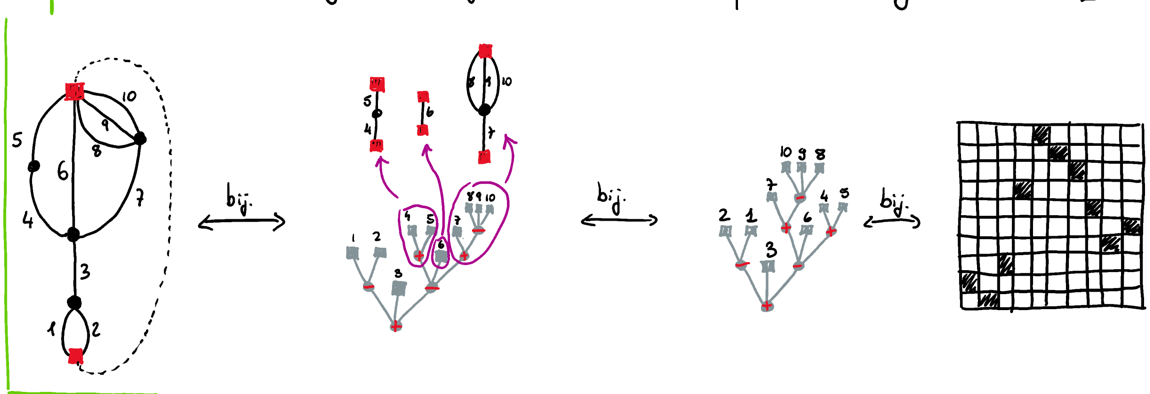
- Series-parallel maps with n edges;
- Schröder trees with n leaves;
- Separable permutations of size n.

(see Wikipedia / OEIS)

$\sim C \cdot \frac{(3+2\sqrt{2})^n}{n^{3/2}}$

Moreover, all these objects are counted by the Large Schroder number S_{n-1}.

Example: [The details of these bijections will be explained during the lecture]



1.2 - Monotone meanders & Baxter permutations

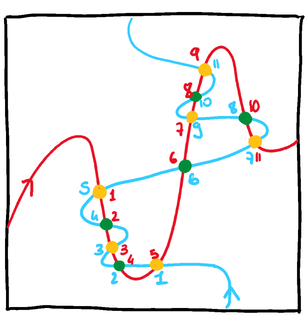
Def: A MONOTONE MEANDER of size n is a pair of simple curves ℓ_1 and ℓ_2 in $[0,1]^2$ which cross each other exactly $2n-1$ times and such that:

- ℓ_1 starts on the left-hand side of $[0,1]^2$, ends on the right-hand side of $[0,1]^2$ and never moves in the left direction. (Equivalently, it is the graph of a cont. function from $[0,1]$ to $[0,1]$.)

- ℓ_2 starts on the bottom side of $[0,1]^2$, ends on the top side of $[0,1]^2$ and never moves in the bottom direction. (Equivalently, it is the graph of a cont. function from $[0,1]$ to $[0,1]$ rotated by 90 degrees).

We identify two monoton meanders (ℓ_1,ℓ_2) & (ℓ_1',ℓ_2') if $\exists\ \gamma : [0,1]^2 \to [0,1]^2$ homeomorphism s.t. $\gamma(\ell_1)=\ell_1'$, $\gamma(\ell_2)=\ell_2'$.

Example: There is a natural way to encode a monotone meander with a permutation:



Read the red labels following the blue path.

$$\longrightarrow \sigma = 5\ 4\ 3\ 2\ 1\ 6\ 9\ 10\ 11\ 8\ 7$$

Reduce the yellow permutation to the unique permutation on $\{1,\dots,n\}$ preserving the relative order between numbers.

$$\xrightarrow{\varphi} \pi = 3\ 2\ 1\ 5\ 6\ 4$$

"Def": The set φ(Monotone meanders) is called the set of Baxter permutations.

Proposition: The map φ is a bijection between monotone meanders of size n and Baxter permutations of size n.

Baxter permutations enjoys several nice combinatorial properties & equivalent definitions:

Def: A BAXTER PERMUTATION is a permutation avoiding the patterns 2-41-3 and 3-14-2

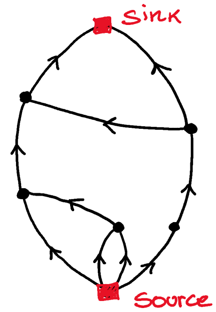
i.e. σ is Baxter $\iff \nexists\ i_1<i_2<i_3$ s.t. $\sigma(i_2+1)<\sigma(i_1)<\sigma(i_3)<\sigma(i_2)$ or $\sigma(i_2)<\sigma(i_3)<\sigma(i_1)<\sigma(i_2+1)$

Also Baxter permutations (as separable permutations) have a very reach combinatorial structure, and they are known to be in bijection with many other combinatorial structures. For instance, bipolar oriented planar maps.

Def: A BIPOLAR ORIENTATION of a planar map M is an orientation of the edges such that:

1) There are no oriented cycles;
2) There is exactly one source (vertex with only outgoing edges);
3) There is exactly one sink (vertex with only incoming edges).

Example:



[Note that we can always draw the map with the edges oriented from bottom to top.]

THEOREM: [Bonichon, Bousquet-Mélou, Fusy, 2008]
The following objects are in bijection:

- Baxter permutations of size n;
- Bipolar orientations with n edges.

Moreover, all these objects are counted by the Baxter number B_n. $\sim C \cdot \frac{8^n}{n^4}$

Question: How does a large random uniform Baxter/Separable permutation look like?

2- Permutons: The natural scaling limits of permutations.

Definition: A PERMUTON μ is a probability measure on the unit square $[0,1]^2$ with uniform marginals, i.e. $\mu([a,b]\times[0,1]) = \mu([0,1]\times[a,b]) = b-a \quad \forall\, 0\leq a\leq b\leq 1.$

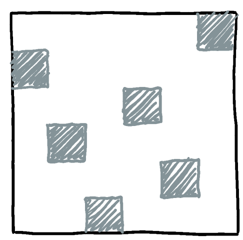
There is a natural way to associate a permutation with a permuton.

Example: If

$$\sigma = 5\,3\,1\,4\,2\,6 \qquad \text{(one-line notation)}$$

then

$$\mu_\sigma =$$



(Lebesgue measure on the gray support of total mass 1)

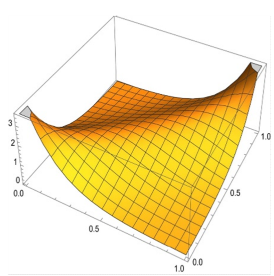
We then have a natural notion of convergence for permutations:

$$\sigma_n \xrightarrow{n\to\infty} \mu \iff \mu_{\sigma_n} \xrightarrow{n\to\infty} \mu \quad \text{w.r.t. weak-topology.}$$

Some interesting permutons:

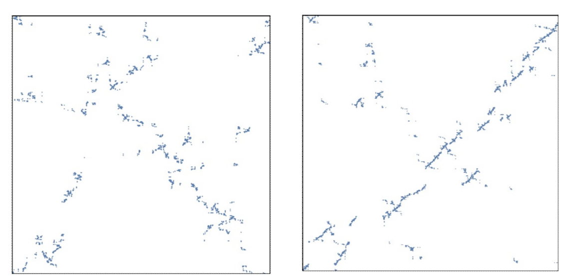
Uniform

[Permuton limit of uniform permutations]



Mallows

[Permuton limit of Mallows permutations]



Pattern-avoiding

[As you can see, sometime the permuton limit can be VERY irregular]

A BIT of HISTORY:

- Introduced by Hoppen, Kohayakawa, Moreira, Roth, Sampaio in 2011 to study permutation sequences and parameter testing (as a counterpart of graphons for graph).

- Popularized in the probabilistic community by:

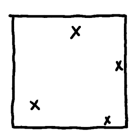
  * Kenyon, Kral, Radin, Winkler ⇝ Permutations with fixed pattern densities.
  * Bassino, Bouvel, Féray, Gerin, Maazoun, Pierrot ⇝ Limits of pattern-avoiding permutations

- There is now a quite large literature studying limits of non-uniform random permutations (see, for instance, my PHD thesis for an overview).

Some remarks:

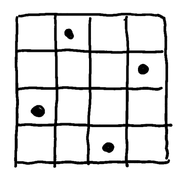
- Permuton convergence is equivalent to the convergence of the proportion of permutation patterns
- We generalized the theory of permutons to high-dimensional analogues
  [Ongoing project with Andrew Lin (Phd student at Stanford)]
- Given a permuton μ on can sample a permutation of size n:

First sample n iid points in $[0,1]^2$ with distribution μ



⇝

Then rescale everything to the diagram of the unique permutation that preserves the order



$= \sigma = 2413$

⚠ If μ is random, you first need to sample μ and then do the operation above!

3- Permuton convergence for Baxter & Separable permutations

THEOREM: (Bassino, Bouvel, Féray, Gerin, Pierrot, 2017)

Let σ_n be a uniform Separable permutation of size $n \in \mathbb{N}$. Then

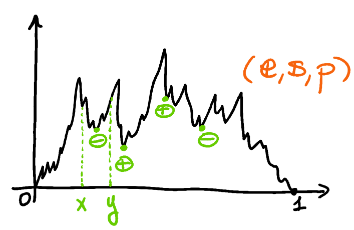
$$\mu_{\sigma_n} \xrightarrow[n\to\infty]{d} \mu^S_{1/2} \rightsquigarrow \text{The Brownian Separable Permuton of param. } \tfrac{1}{2}$$

Question: What is $\mu^S_{1/2}$?

3.1 - The Brownian Separable permutons

The Brownian Separable permutons $(\mu^S_p)_{p\in[0,1]}$ are a one-parameter family of random permutons.

Fix $p \in [0,1]$. It can be constructed as follows:



$\Rightarrow x \triangleleft y$

- Consider a Brownian excursion $\mathbf{e} = (\mathbf{e}_t)_{t\in[0,1]}$.
- Consider an independent collection $\mathcal{S}$ of iid $\oplus/\ominus$-signs attached to each local minimus of $\mathbf{e}$ such that

$$\mathbb{P}(\oplus) = 1 - \mathbb{P}(\ominus) = p$$

- For each $x < y \in (0,1)$ we say that $x \triangleleft y$ iff $\mathcal{S}_{x,y} = \oplus$, where

$$\mathcal{S}_{x,y} = \mathcal{S}\left(\min_{t\in[x,y]} \mathbf{e}(t)\right)$$

- We get that $\triangleleft$ is a RANDOM TOTAL ORDER (on a subset of $[0,1]$ of Leb-meas. 1)

A little intermezzo:

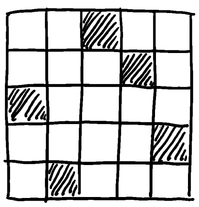
- Given a TOTAL ORDER $\preccurlyeq$ on $\{1,\dots,n\}$ one can construct a PERMUTATION:

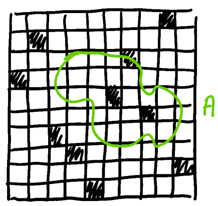
$$\sigma(i) := \#\{j \mid j \preccurlyeq i\}$$

Example: $2 \preccurlyeq 5 \preccurlyeq 1 \preccurlyeq 4 \preccurlyeq 3$

$$\Rightarrow \sigma = 3\,1\,5\,4\,2$$

- One can associate a PERMUTON to a PERMUTATION:

$\mu_\sigma =$ 



(Taking a larger n, we get a finer grid)

- $\mu_\sigma(A) \approx \frac{1}{n} \# \{i \mid (i,\sigma(i)) \in A\}$

The continuum analogue:

- Given the TOTAL ORDER $\triangleleft$ on $[0,1]$, one can contruct a "CONTINUUM PERMUTATION":

$$\varphi(x) := \mathrm{Leb}\{y \in [0,1] \mid y \triangleleft x\} \quad \forall x \in [0,1]$$

- One can associate a PERMUTON to φ:

$$\mu_\varphi(A) = \mathrm{Leb}\big(\{t \in [0,1] \mid (t, \varphi(t)) \in A\}\big)$$

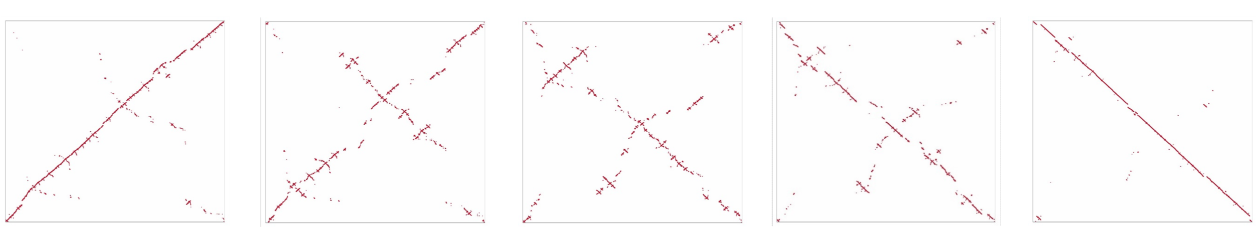
i.e. $\mu_\varphi = (\mathrm{Id}, \varphi)_* \mathrm{Leb}|_{[0,1]}$

Definition: If $\triangleleft$ is the total order constructed above from (e, s, p) then

$\mu_p^S := \mu_\varphi$ is the BROWNIAN SEPARABLE PERMUTON (BSP) with parameter p.

Important fact: μ_p^S is RANDOM, i.e. is a random probability measure. Equivalently, you can think of μ_p^S as a random variable in the space of permutons.

Some simulations:



$p=0.9$ $\quad$ $p=0.6$ $\quad$ $p=0.5$ $\quad$ $p=0.4$ $\quad$ $p=0.1$

Remarks:

- The Brownian Separable permuton has been shown to be a universal limiting permuton, i.e. it describes the permuton limit of many different families of pattern-avoiding permutations. For more details, see for instance:

  ① "Universal limits of substitution-closed permutations classes" by Bassino, Bouvel, Feray, Gerin, Maazoun, Pierrot (2019)

  ② "A decorated tree approach to random permutations in substitution-closed classes" by B., Bouvel, Féray, Stufler (2020)

- Several properties of the geometry of the Brownian Separable permutons have been studied by M. Maazoun in the paper "On the Brownian separable permuton".

3.2 - The Baxter permuton

Theorem: [B., Maazoun, 2021]

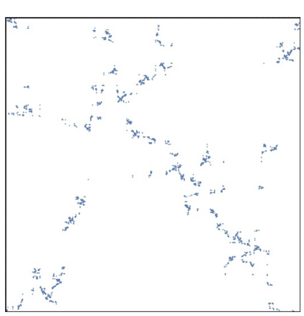
If σ_n is a uniform Baxter permutation of size n, then

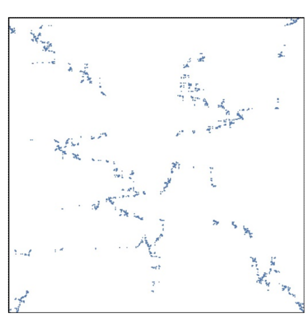
$$\mu_{\sigma_n} \xrightarrow[n\to\infty]{d} \mu_B =: \text{Baxter permuton}$$

Remark: The Baxter permuton μ_B is a random permuton.

Simulations:

Baxter permutations





IMPORTANT COMMENSTS:

- The proof of the theorem above involves the study of certain <u>coalescent-walk</u> <u>processes</u>.

- The Baxter pemuton can be constructed from a <u>two-dim. correlated Brownian exc.</u>
 +
 <u>a system</u> of <u>SDEs</u>.

- The Baxter permuton μ_B is <u>different</u> in law from the Brownian Separable permuton μ_p^S, $\forall p \in (0,1)$.

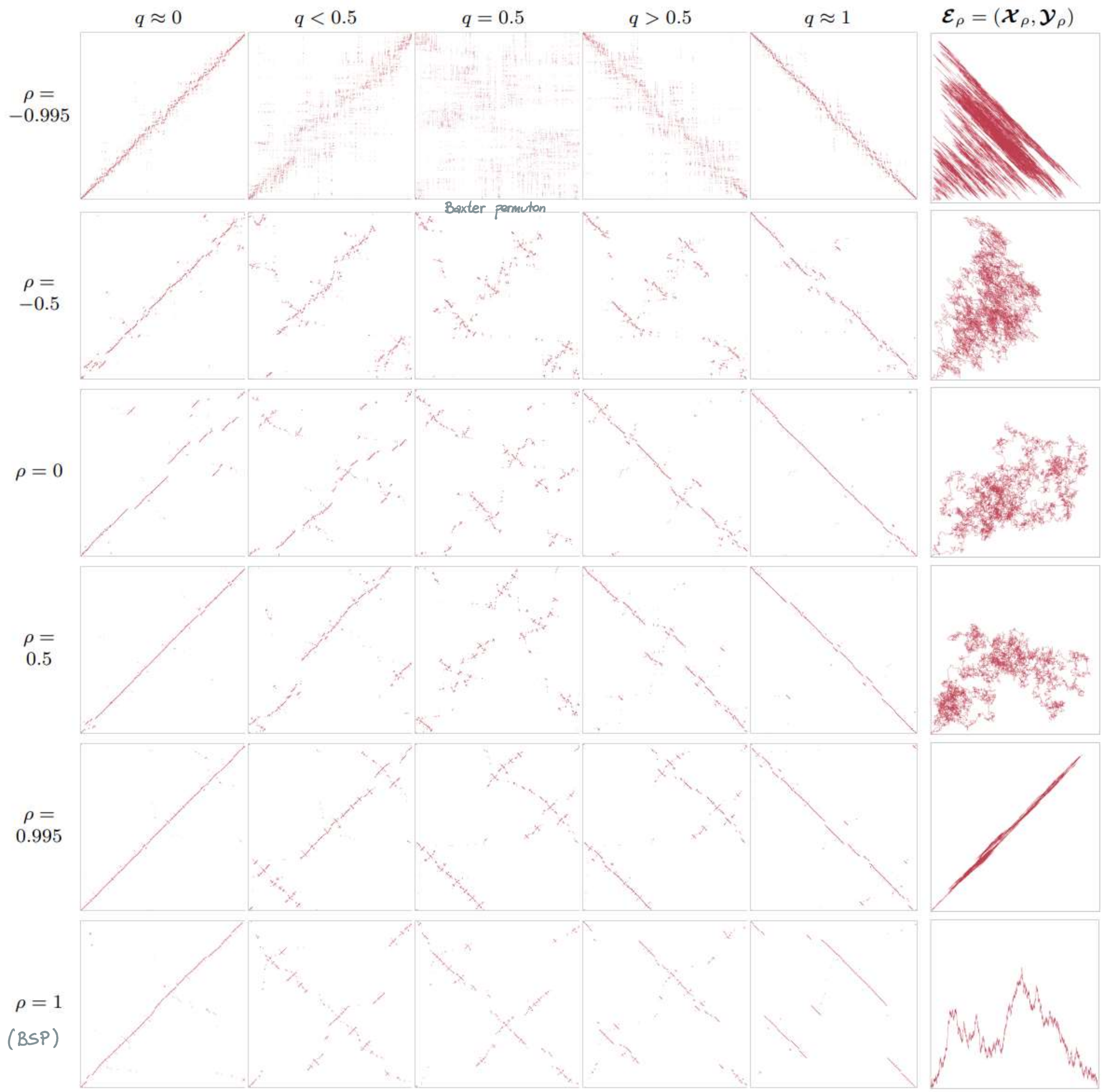
3.3 - The skew Brownian permuton

The fact that $\mu_B \neq \mu_p^S$, $\forall p \in (0,1)$, motivated me to try to study the connection between these permutons.

I recently introduced a two-parameter family $(\mu_{\rho,q})_{\rho\in[-1,1],\, q\in[0,1]}$ of random permutons, called <u>skew Brownian permutons</u> which are <u>universal objects</u> (i.e. scaling limits of many models of random permutations) and such that

$$\mu_{-\frac{1}{2},\frac{1}{2}} \stackrel{d}{=} \mu_B \quad \& \quad \mu_{1,q} \stackrel{d}{=} \mu_{1-q}^S \quad \forall q \in [0,1]$$

Simulations :



Some comments:

- The permutons $\mu_{\rho,q}$ are constructed starting from a 2D-Brownian excursion with correlation $\rho \in (-1,1]$ and solving a system of SDEs (which depends on $q \in [0,1]$).

- In [B., 2023], I proved that when $\rho \in (-1,1)$ and $q \in [0,1]$ then

$\mu_{\rho,q}$ can be constructed from
  - two coupled space-filling SLE_κ
  - one independent γ-LQG.

- LQG and SLEs are certain universal random objects that describes (among many other things.) the scaling limit of various models of random planar maps decorated by a random curve.

- In an ongoing project with E. Gwynne, we proved that

$\mu_{\rho,q}$ determines both SLE_κ + the γ-LQG.

- Many interesting quantities that can be read from the permuton have a geometric interpretation on SLEs and LQG. In this mini-course we will focus on the specific case of the

LONGEST INCREASING SUBSEQUENCE

for the Brownian Separable permuton μ^S_p and its connections with the

LONGEST DIRECTED PATH

in series-parallel maps.

- At the end of the course, if time permits, I will informally discuss what should be the continuum analogue of the quantities that we will study at the discrete level.

[For the experts, it should be a sort of "LQG-directed landscape", i.e. the "quantum" analogue of the directed landscape constructed by Dauvergne, Ortmann and Virag, in the setting of last-passage percolation.]