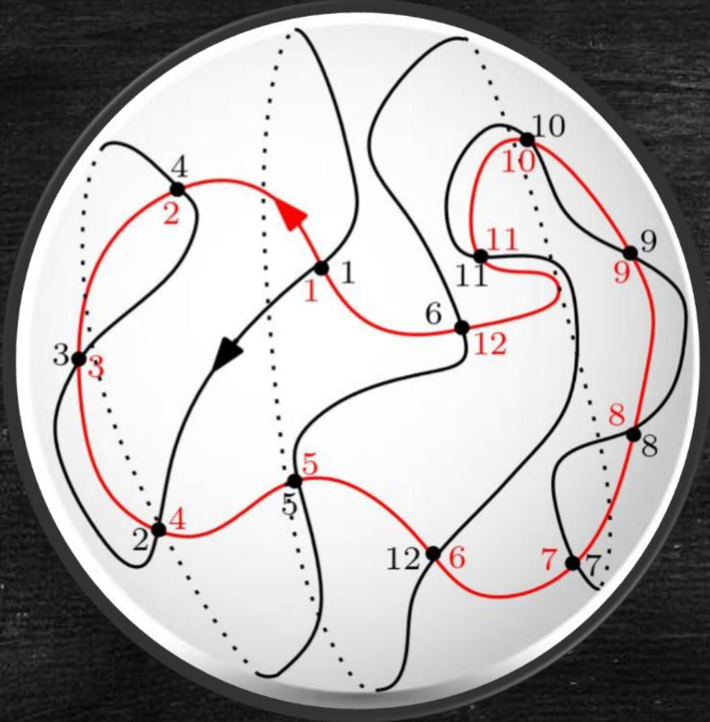
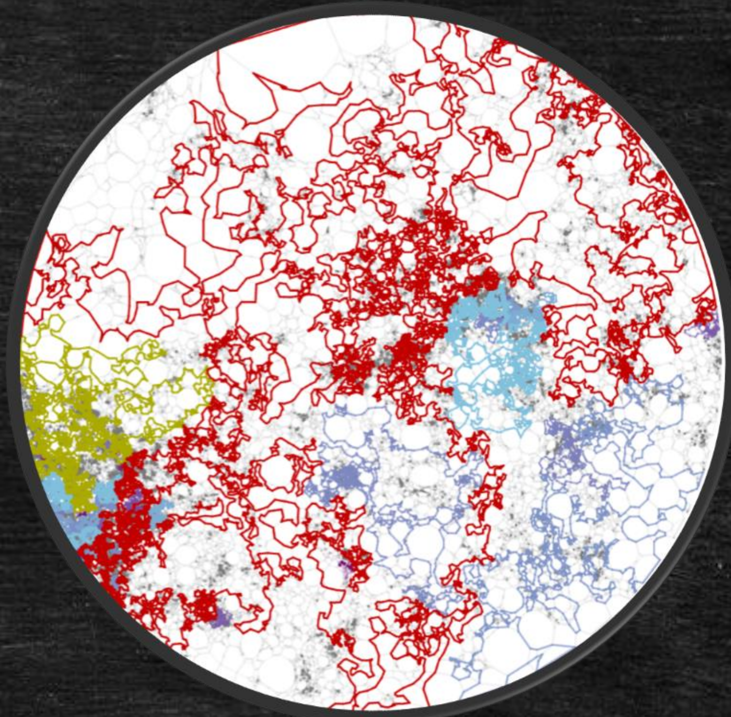




Meanders & Meandric Systems

(joint works with E. Gwynne, M. Park, and X. Sun)

Jacopo Borga

Stanford University

UCLA department colloquium, March 9, 2023

Plan of the talk

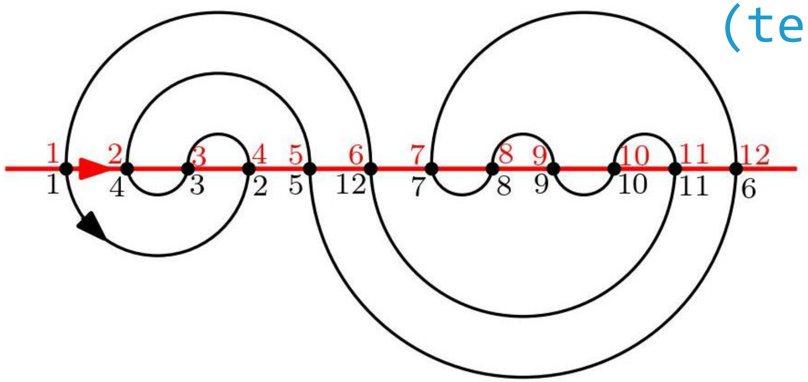
- MEANDERS: A new conjecture & some results
- PERMUTONS, SLE & LQG
- MEANDRIC SYSTEMS: - Some new conjectures
  - Main results

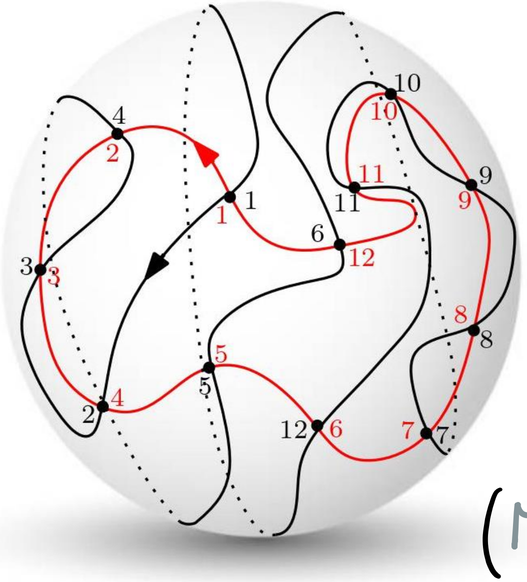
1912

"In how many different ways a simple loop in the plane can cross a line a specified number of times?"

Meanders

(term coined by V.Arnold)





$\sigma_n = 1\ 4\ 3\ 2\ 5\ 12\ 7\ 8\ 9\ 10\ 11\ 6$

(M_n, P_n^1, P_n^2)

LITERATURE: • Zvonkin, 2021, "Meanders: A personal perspective"
(Survey papers) • La Croix, 2003, "Approaches to the enumerative theory of meanders"

Connections to many different subjects: COMBINATORICS, THEORETICAL PHYSICS, GEOMETRY of MODULI SPACES, ...

WHAT DOES A LARGE UNIFORM RANDOM MEANDRIC PERMUTATION LOOK LIKE?

Just sampling a UNIFORM MEANDER of large fixed size is QUITE HARD.

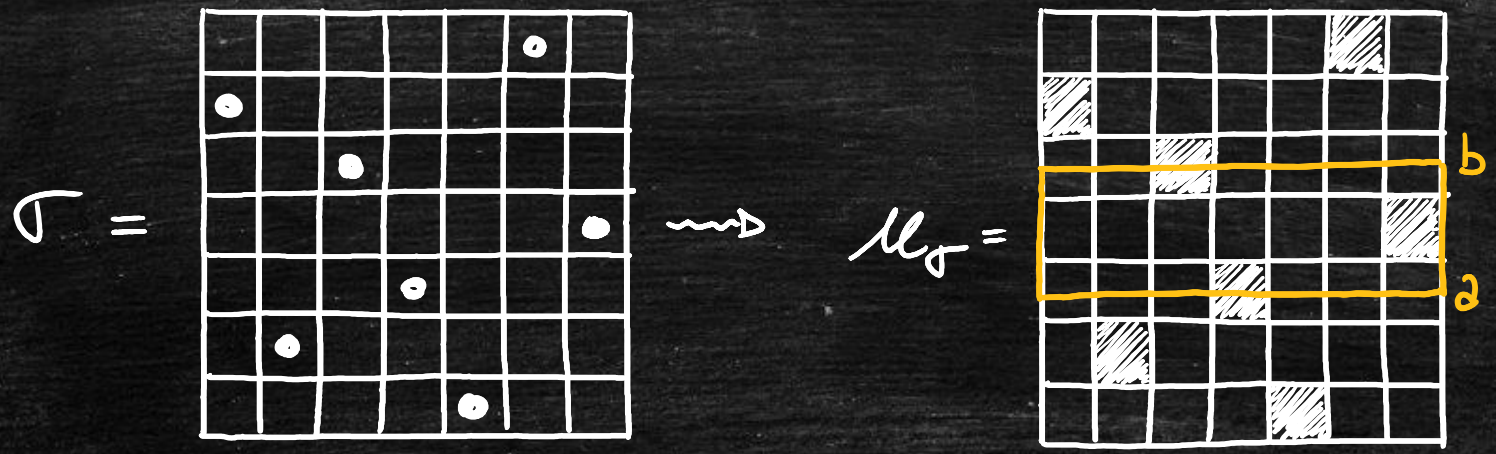
OUR GOAL: IDENTIFY and STUDY the scaling limit of large uniform random meandric permutations

Our techniques introduced in the rest of the talk are much more general

Permutons,
Liouville quantum gravity
&
Schramm-Loewner evolutions

Permutons

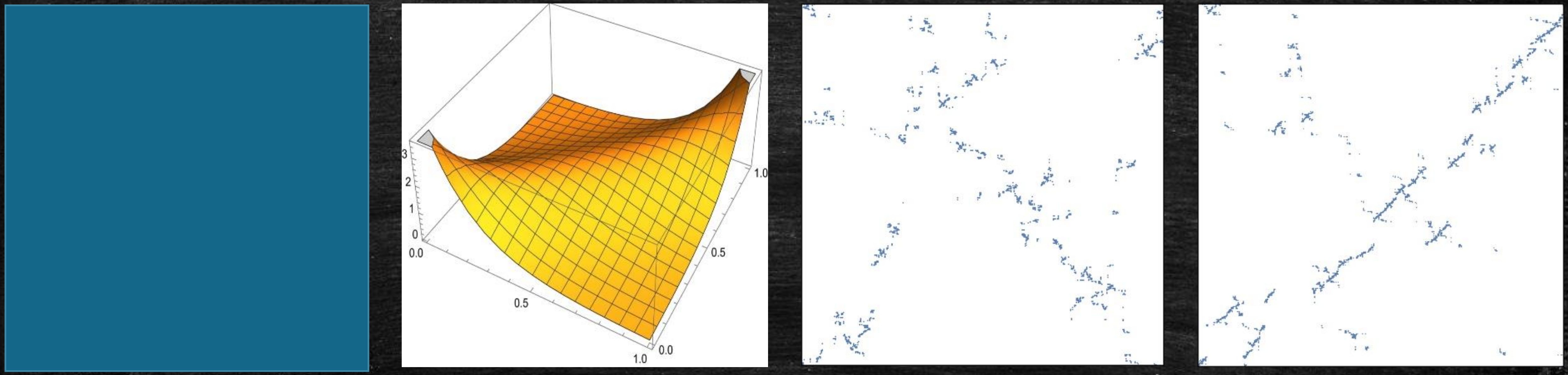
Consider the permutation $\sigma = 6\ 2\ 5\ 3\ 1\ 7\ 4$



Definition: A PERMUTON is a probability measure on $[0,1]^2$ with uniform marginals.

Definition: A PERMUTON is a probability measure on $[0,1]^2$ with uniform marginals.

Examples:



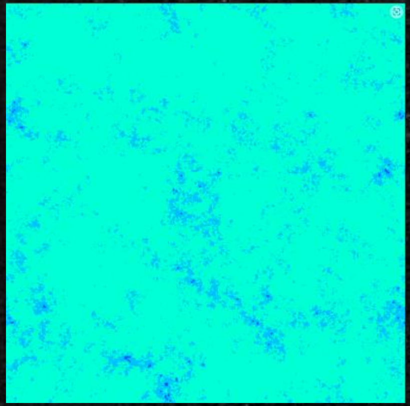
- It is an emerging but rapidly expanding topic.
- Let $(\sigma_n)_n$ be a sequence of permutations & $(\mu_{\sigma_n})_n$ are the corresponding permutons:

$$\mu_{\sigma_n} \xrightarrow{\text{weakly}} \mu \implies \frac{\#\text{ of inversions of } \sigma_n}{n^2} \longrightarrow inv(\mu).$$

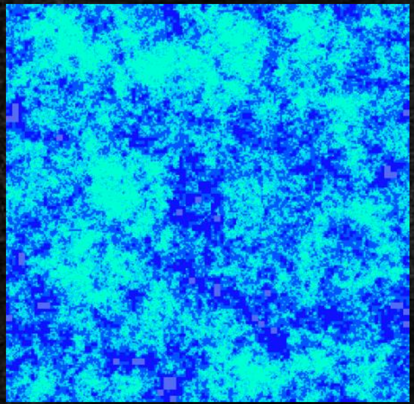
MORAL: Permutons convergence implies convergence of some classical permutation statistics.

Liouville quantum gravity

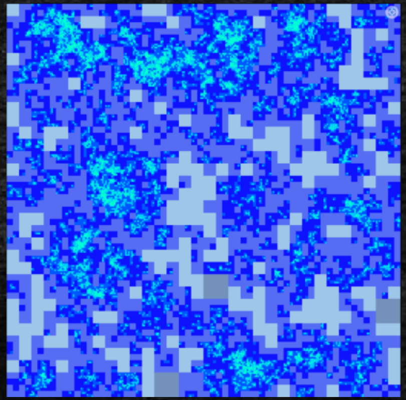
- Fix $\gamma \in (0,2)$ [This parameter controls the "roughness" of the measure]
- A γ-LQG μ is a "natural" RANDOM PROBABILITY MEAUSURE on the Riemann sphere $\hat{\mathbb{C}}$, i.e. $\mu(\hat{\mathbb{C}}) = 1$. Moreover, almost surely, μ is non-atomic, i.e. $\mu(\{x\}) = 0$ $\forall x \in \hat{\mathbb{C}}$, and μ is positive on every open set.



$\gamma \approx 0$



$\gamma \approx 1$

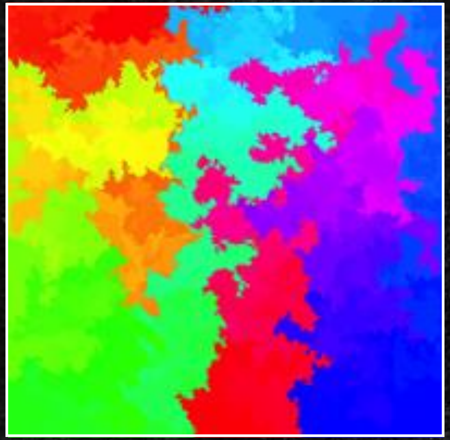


$\gamma \approx 2$

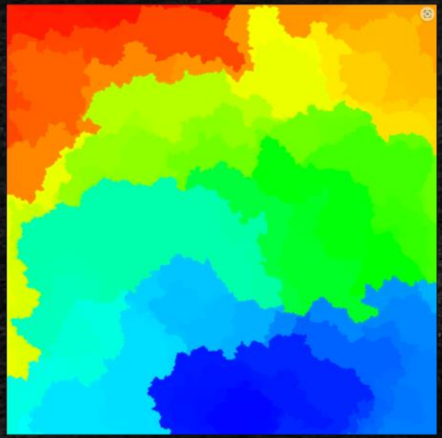
- It is "natural" in many ways, but for today it is "natural" because it is the scaling limit of many models of random planar maps.

Schramm-Loewner evolutions

- Fix $\kappa > 4$. [This parameter controls the "roughness" of the curve]
- A κ- WHOLE-PLANE SPACE-FILLING SLE η is a "natural" RANDOM SPACE-FILLING CURVE on $\hat{\mathbb{C}}$ starting at ∞ and ending at ∞.
  - It is "natural" in many ways, but for today it is "natural" because it is the scaling limit of many interfaces of statistical mechanics models (both on deterministic lattices or random planar maps)



$\kappa=6$



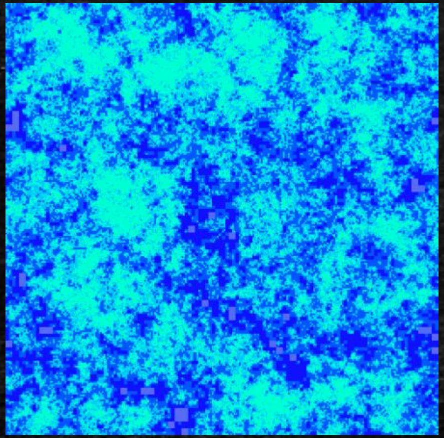
$\kappa=16$

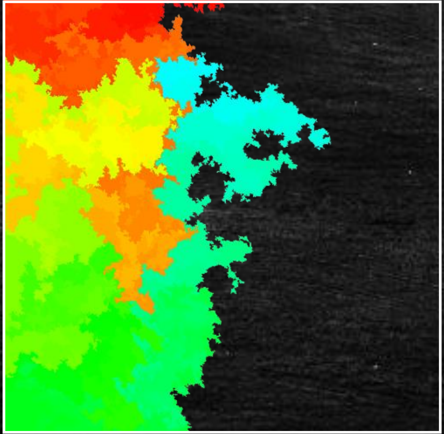
The time parametrization

- Let μ be a γ-LQG AREA MEASURE with $\mu(\hat{\mathbb{C}}) = 1$. $\gamma \in (0,2)$

$\perp\!\!\!\perp$ [A random PROBABILITY MEASURE on $\mathbb{C}$; non-atomic; assigns positive mass to open subsets]

- Let η be a WHOLE-PLANE SPACE-FILLING SLE on $\hat{\mathbb{C}}$. $\kappa > 4$





$\eta(1/2)$

- Parametrize η by μ, i.e., $\mu(\eta([0,t])) = t$, for all $t \in [0,1]$.

Permutons from Liouville quantum gravity & Schramm-Loewner evolutions

The recipe

The ingredients:

- Fix $\gamma \in (0,2)$ and $\kappa_1, \kappa_2 > 4$.
- Let μ be a γ-LQG area measure with $\mu(\hat{\mathbb{C}}) = 1$.
- Let (η_1, η_2) be a pair of space-filling SLEs of parameters (κ_1, κ_2).

  ↳ coupling is unspecified.

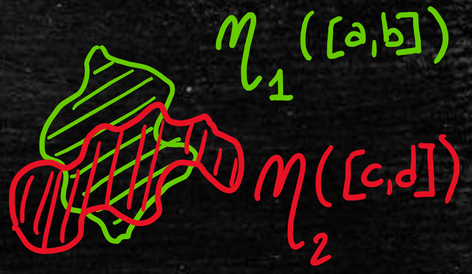
Directions:

- For $i \in \{1,2\}$, parametrize η_i by μ.

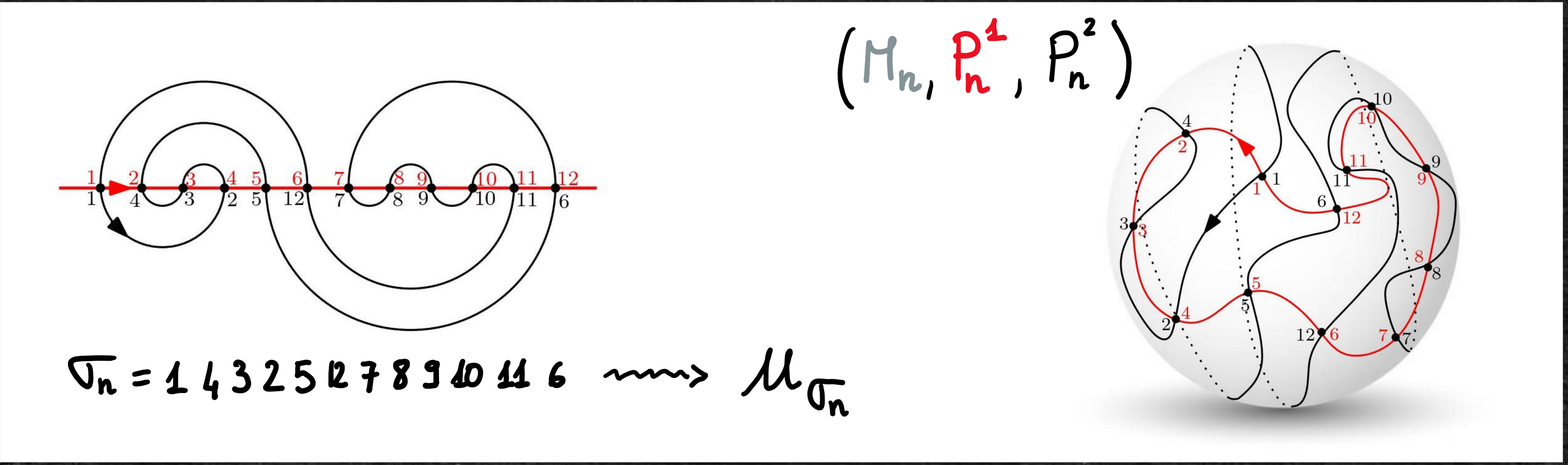
This is a <u>RANDOM PERMUTON</u> & its law depends on $(\gamma, \kappa_1, \kappa_2)$ and the coupling of (η_1, η_2)

The permuton π associated with (μ, η_1, η_2) is defined by

$$\pi\big([a,b]\times[c,d]\big) = \mu\Big(\eta_1([a,b]) \cap \eta_2([c,d])\Big)$$



The meander conjecture



MEANDRIC PERMUTON (MP): $\kappa_1 = \kappa_2 = 8$, $\gamma = \sqrt{\frac{1}{3}\left(17 - \sqrt{145}\right)}$, $\eta_1 \perp\!\!\!\perp \eta_2$

CONJECTURE
(B., Gwynne, Sun, '22)

$$\mathcal{M}_{\sigma_n} \xrightarrow{d} \text{MP} \quad \left(\begin{array}{c}\text{w.r.t weak}\\ \text{topology}\end{array}\right)$$

⚠ The γ-LQG and the two $\perp\!\!\!\perp$ SLE_8 are a.s. given by MP.

CONJECTURE
(B., Gwynne, Sun, '22)

$$(M_n, P_n^1, P_n^2) \xrightarrow{d} \gamma\text{-LQG} + 2 \perp\!\!\!\perp SLE_8 \quad \left(\begin{array}{c}\text{w.r.t. gener.}\\ \text{G-H topology}\end{array}\right)$$

CONJECTURE (B., Gwynne, Sun, '22)

$$\mathcal{M}_{\sigma_n} \xrightarrow{d} MP \quad \left(\begin{matrix}\text{w.r.t weak}\\ \text{topology}\end{matrix}\right)$$

$$\boxed{\begin{matrix} K_1 = K_2 = 8 \\ \gamma = \sqrt{\tfrac{1}{3}\left(17-\sqrt{145}\right)} \qquad \eta_1 \perp\!\!\!\perp \eta_2 \end{matrix}}$$

CONJECTURE (B., Gwynne, Sun, '22)

$$(M_n, P_n^1, P_n^2) \xrightarrow{d} \gamma\text{-LQG} + 2 \perp\!\!\!\perp SLE_8 \quad \left(\begin{matrix}\text{w.r.t. gener.}\\ \text{G-H topology}\end{matrix}\right)$$

These conjectures are motivated by work of Di Francesco, Golinelli, Guitter (2000) where the scaling limit of meanders is described as a CFT with $c=-4$.
(Meanders ≡ $O(0\times 0)$-fully packed loop model on planar maps)

The CFT "interpretation" together with the KPZ formula gives:
KPZ → Knizhnik-Polyakov-Zamolodchikov

CONJECTURE: (Di Francesco, Golinelli, Guitter, 2000)

$$\#\{\text{Meanders of size } 2n\} \underset{n\to\infty}{\sim} \text{Cost} \cdot R^n \cdot n^{-\alpha}, \quad \text{with}$$

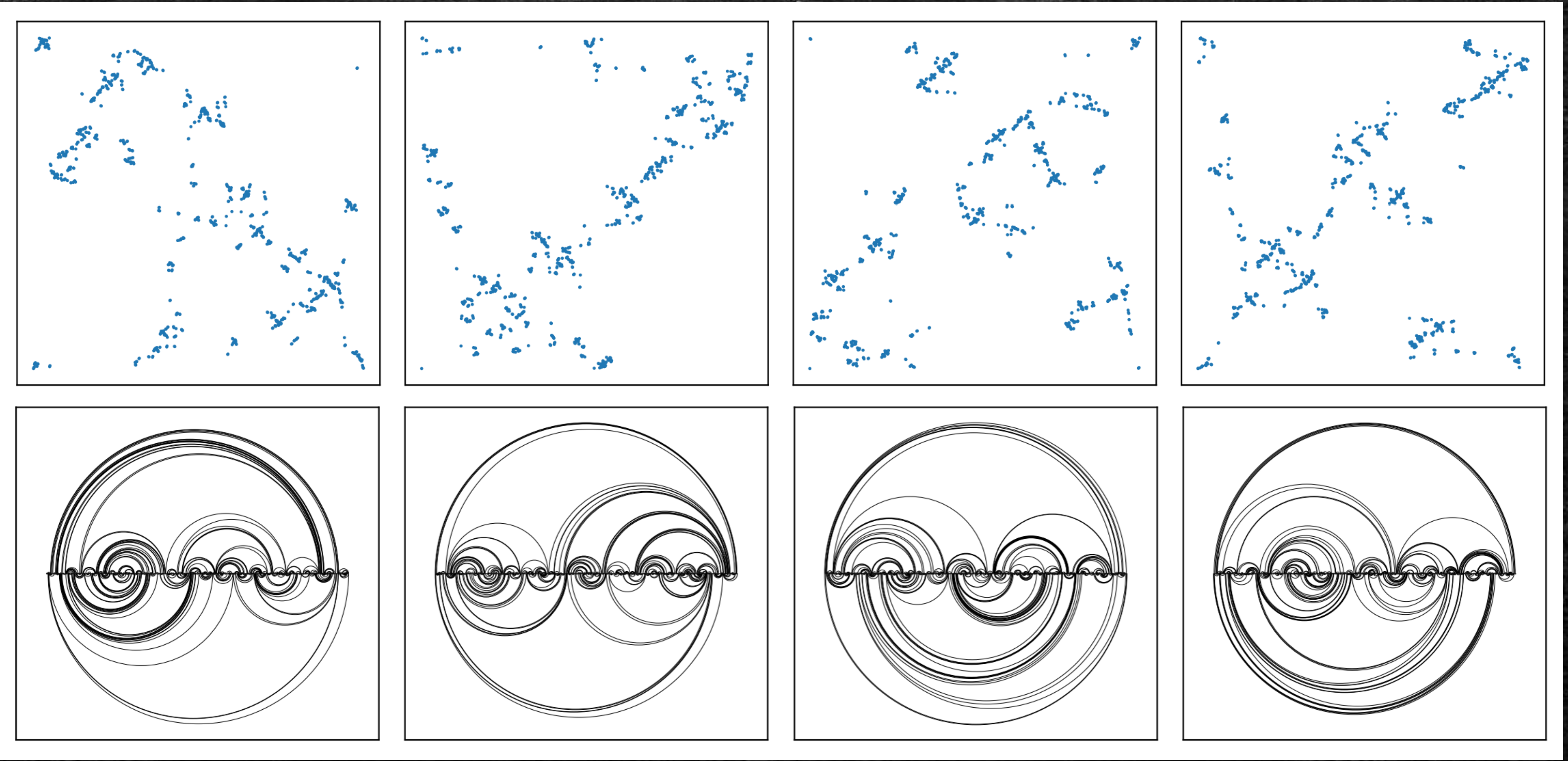
$R \approx 12.26287\ldots$ → Jensen-Guttman

$$\alpha = \frac{29+\sqrt{145}}{12} \approx 3.42$$

The theorems

- Together with E. Gwynne & X. Sun we proved several results on the <u>MEANDRIC PERMUTON</u> using SLE/LQG-techniques.

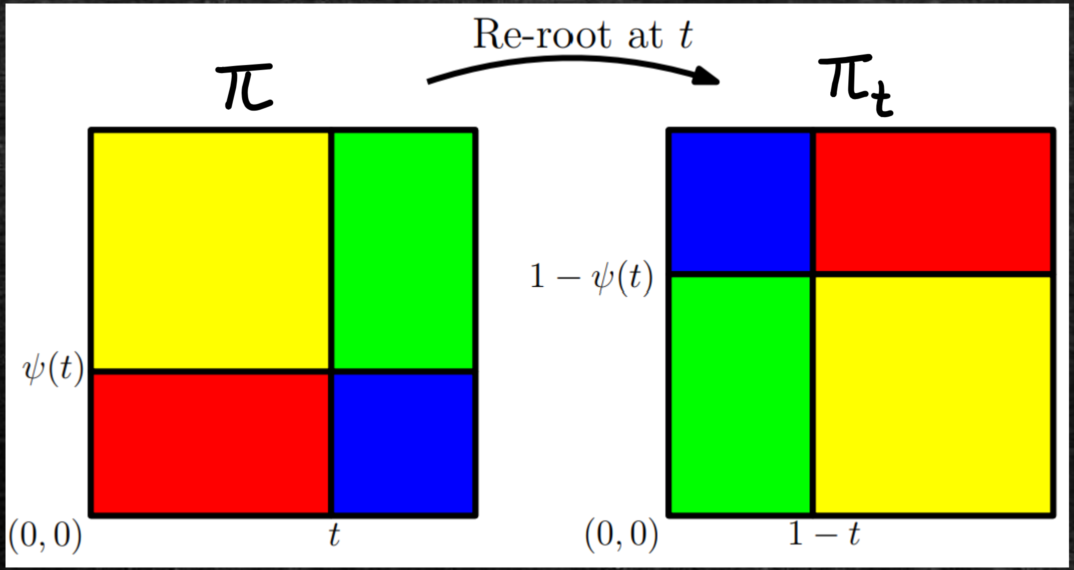
↳ consequences on meandric permutations (if conjecture proved)



@ Zhihan Li

THEOREM: (B., Gwynne, Sun, '22)

Let π be the meandric permuton. Define for $t \in [0,1]$, π_t as follows:



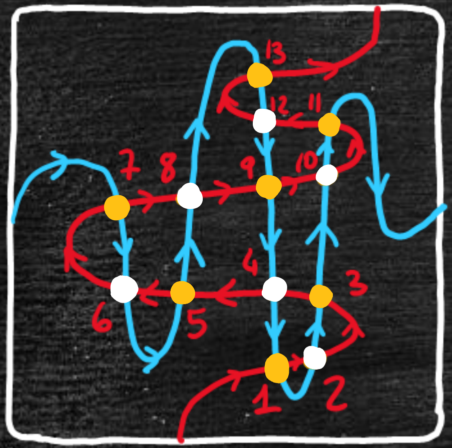
This property is true if and only if $\kappa_1 = \kappa_2 = 8$, but $\forall\, \gamma \in (0,2)$.

Then, for each fixed $t \in [0,1]$, we have

$$\pi \overset{(d)}{=} \pi_t$$

Definition: A MONOTONE MEANDER of size $2n+1$ is:

[Connected with many combinatorial objects]



$$\overset{\psi}{\rightsquigarrow} \quad \sigma := \overbrace{7\;6\;5\;8\;13\;12\;9\;4\;1\;2\;3\;10\;11}^{4\;3\;7\;5\;1\;2\;6 \text{ BAXTER PERMUTATION}}_{\text{MONOTONE MEANDRIC PERMUTATION}}$$

(This is a bijection)

THEOREM (B., Maazoun, AOP '21) + (B., PLMS '23)

Let σ_n be a uniform Baxter permutation of size n, then

$$\mu_{\sigma_n} \xrightarrow{d} \mu_B \rightsquigarrow$$

BAXTER PERMUTON: [IMAGINARY GEOMETRY of Miller & Sheffield]

Obtained from two coupled SLE_{12}

& an independent $\sqrt{4/3}$ – LQG

KEY IDEA:

Baxter permuton is a particular instance

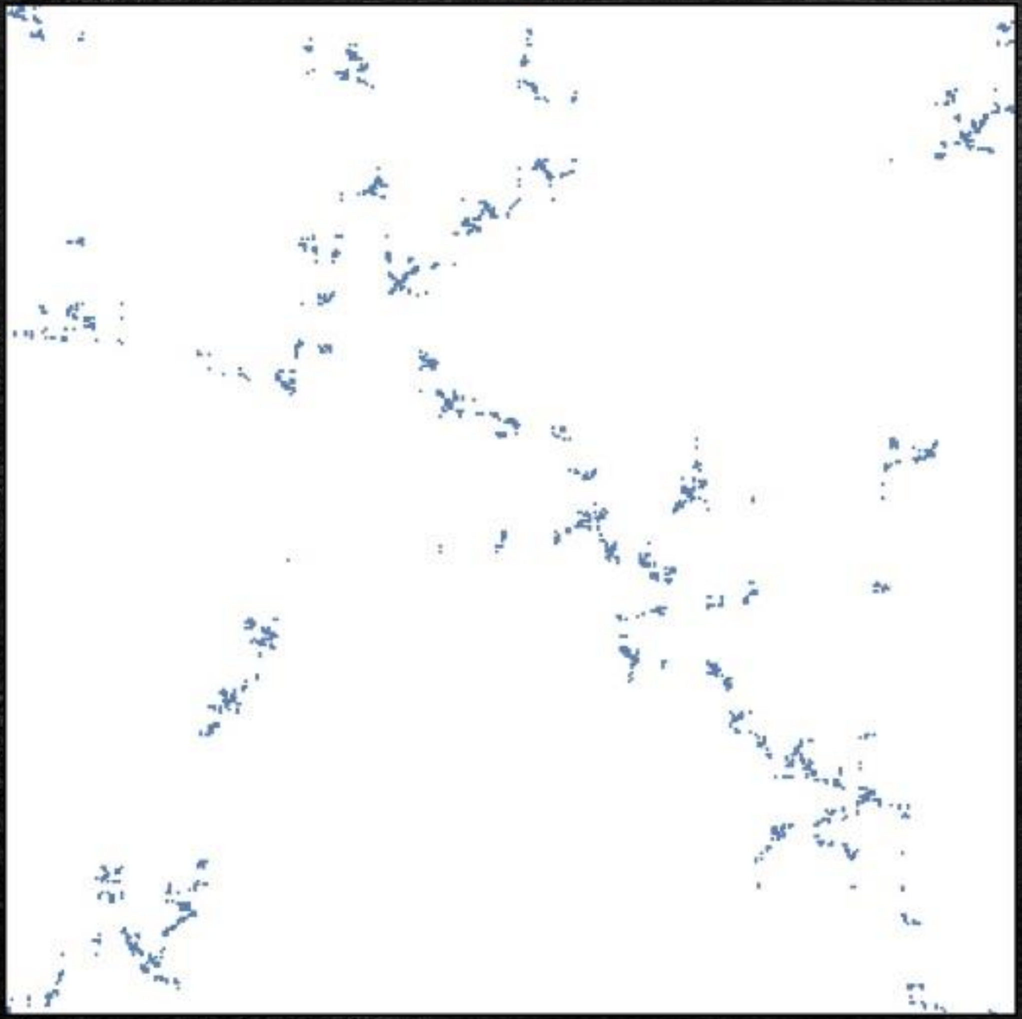
In [B., "The skew-Brownian permuton", PLMS 2023], building on the "MATING OF TREES" of Duplantier, Miller & Sheffield, I explained how to describe the Baxter permuton using:

- CORRELATED 2-DIM BROWNIAN EXCURSIONS;
- Some FLOWS of STOCHASTIC DIFFERENTIAL EQUATIONS.

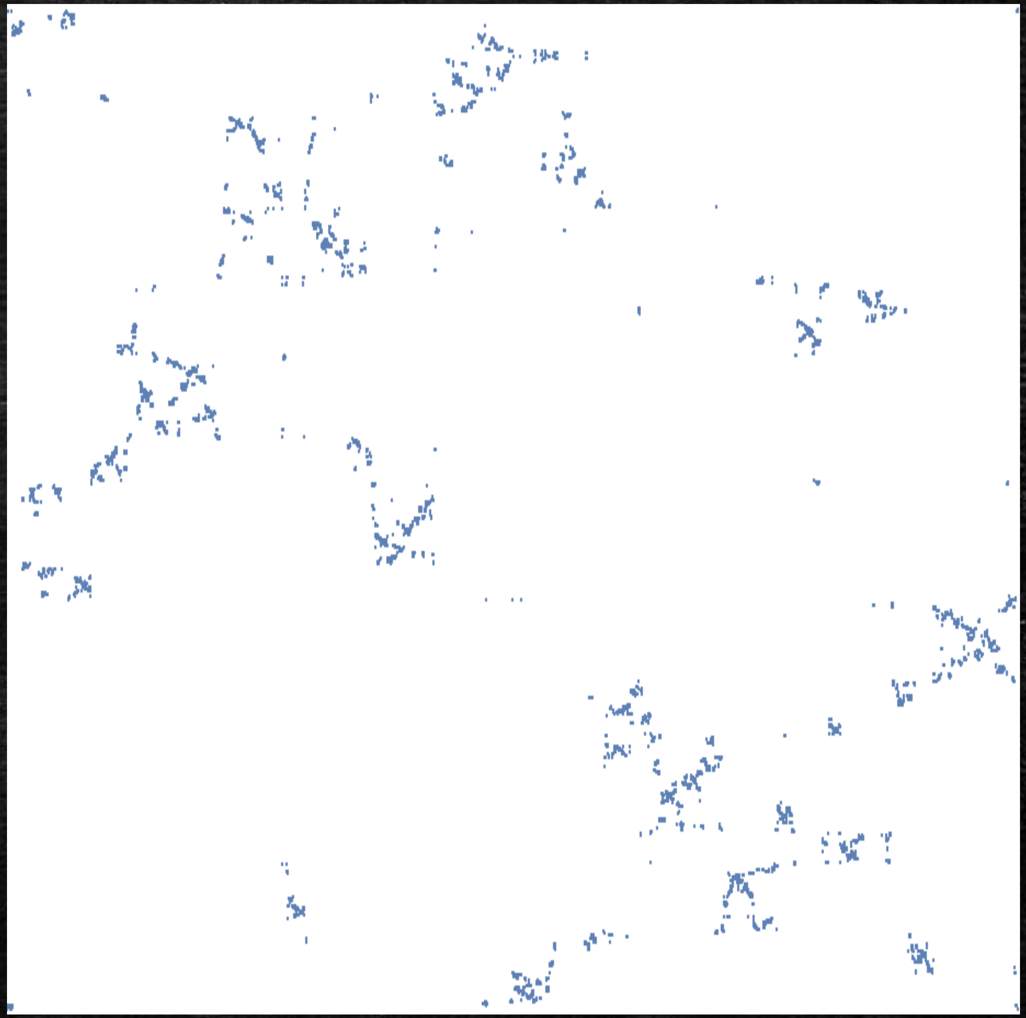
OPEN QUESTION:

- Can we find something similar for the Meandric permuton?
- Can we characterize the Meandric permuton without using SLE & LQG?

Baxter permutation



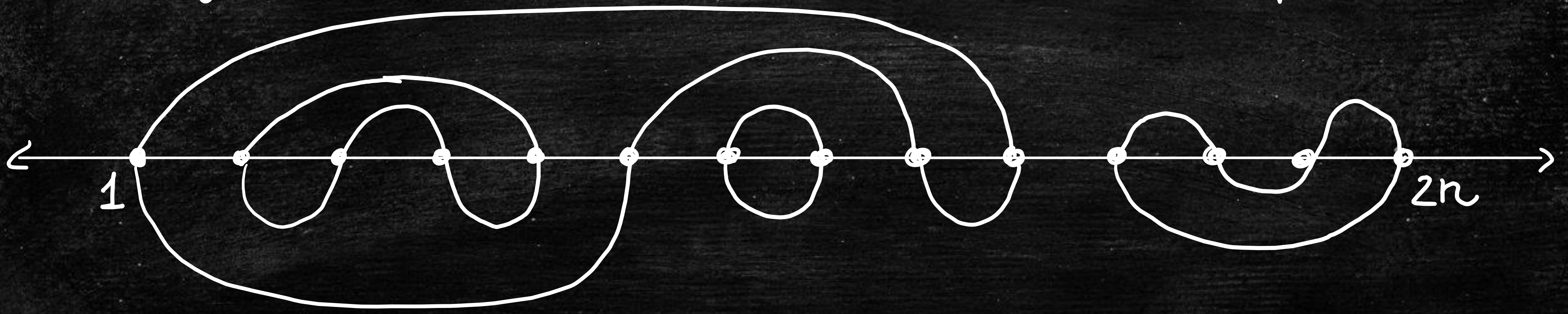
Meandric Permutation



Meandric systems

Meandric systems

Def: A MEANDRIC SYSTEM of size $n \in \mathbb{N}$ is a collection of (many) meanders with total size n. More precisely, it is a collection of disjoint simple loops in $\mathbb{R}^2$ which do not hit $\mathbb{R}$ without crossing it and which cross $\mathbb{R}$ precisely at the points $\{1, \dots, 2n\}$, viewed modulo homeomorphisms fixing $\mathbb{R}$.

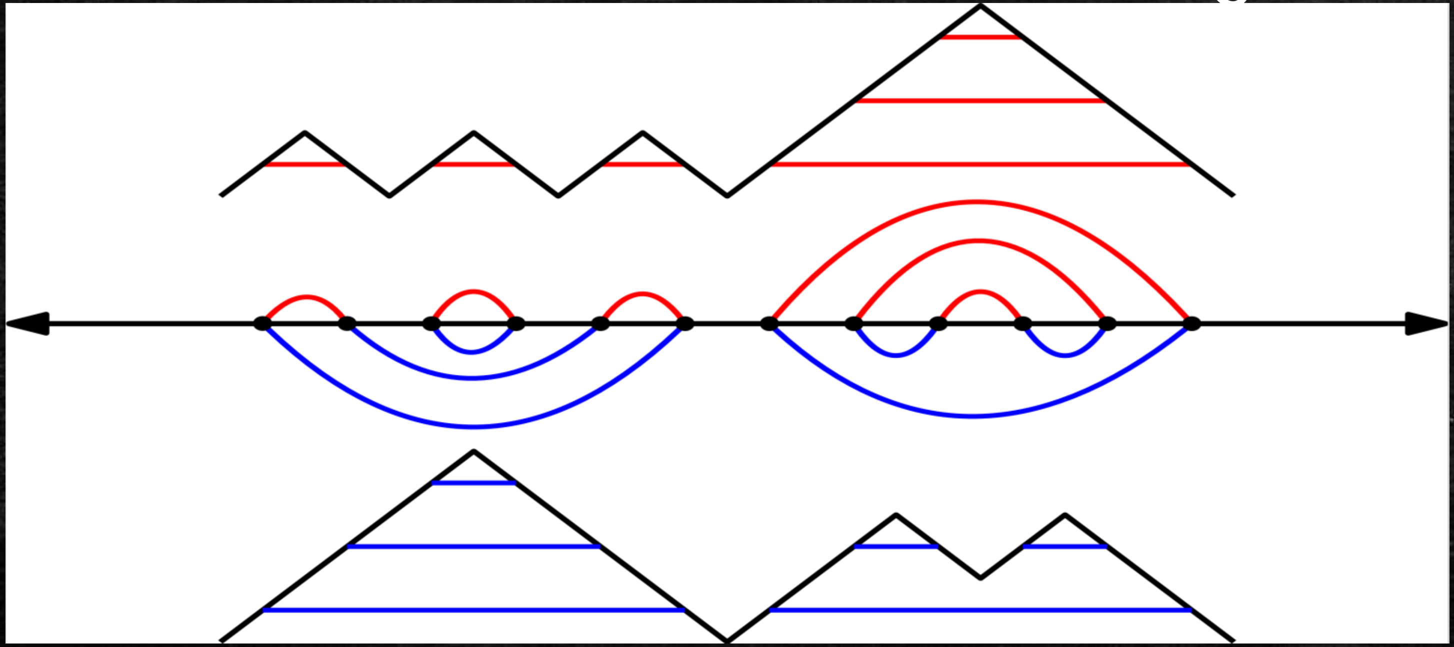


- STUDIED BY : Kargin, Féray-Thévenin, Curien-Kozma-Sidoravicious-Tournier
  Golden-Nica-Puder, Fukuda-Nechita, Janson-Thévenin, etc...

- This model is equivalent to:

  FULLY PACKED $O(0\times 1)$ loop model on PLANAR MAPS

- How can we sample a uniform meandric system of size $2n$?



SOME QUESTIONS: (Loops are a complicated functional of the walks)

1. How many loops?
   - FÉRAY-THÉVENIN (2022, IMRN): $\# \text{loop} \sim c \cdot n$, where c is a complicated sum over meanders.

2. What is the "SIZE" of the largest loop?
   - Kargin (2022): The largest loop contains $\geq c \cdot \log(n)$ vertices
     ↳ Simulations suggest $\approx n^{\alpha}$ with $\alpha \approx 4/5$.

3. Does one loop dominate? Or, are there many large loops of similar "size"?
   - Very related to the "infinite noodle" of Curien-Kozma-Sidoravicius-Tournier
     ↳ CONJ: There is NO INFINITE LOOP

4. What is the scaling limit as $n \longrightarrow \infty$? ???

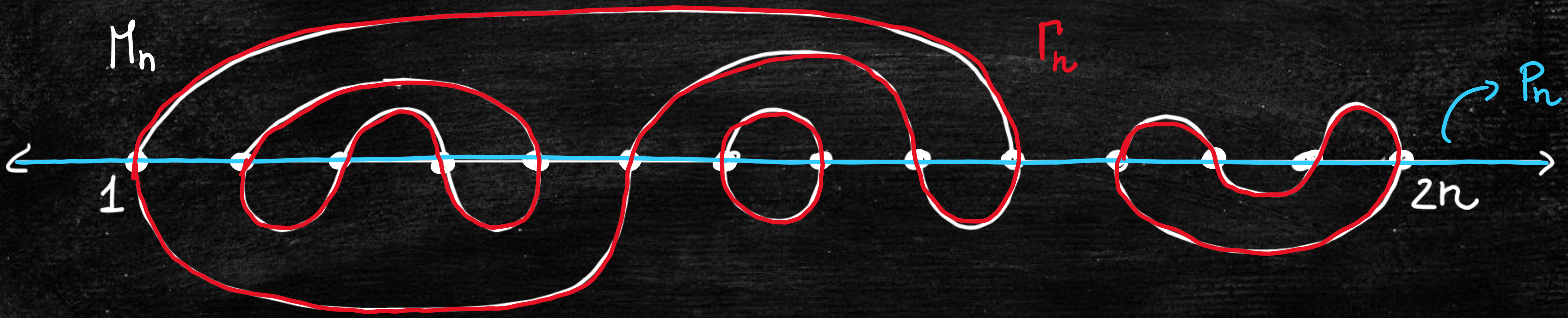
GOAL: • Conjectures for answers to the above questions;

• Rigorous results in the direction of these conjectures.

We view a meandric system as a

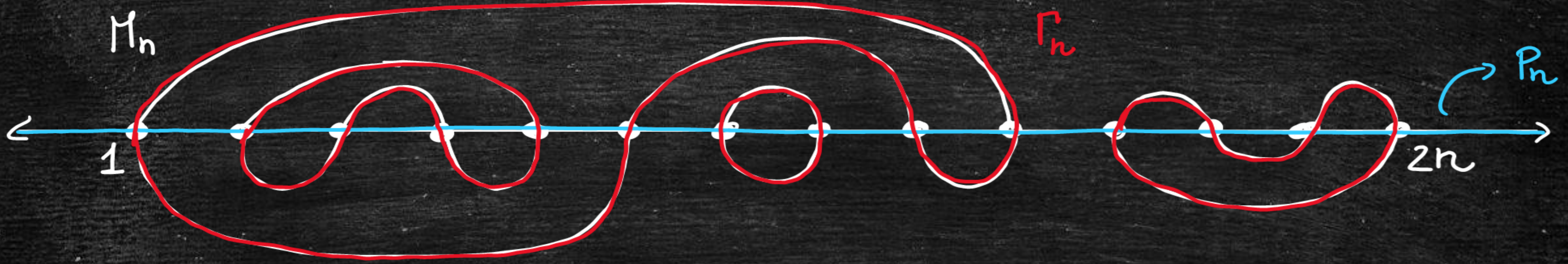
PLANAR MAP + HAMILTONIAN PATH + LOOPS

M_n $\qquad$ P_n $\qquad$ Γ_n



PLANAR MAP + HAMILTONIAN PATH + LOOPS

M_n $\quad P_n$ $\quad \Gamma_n$

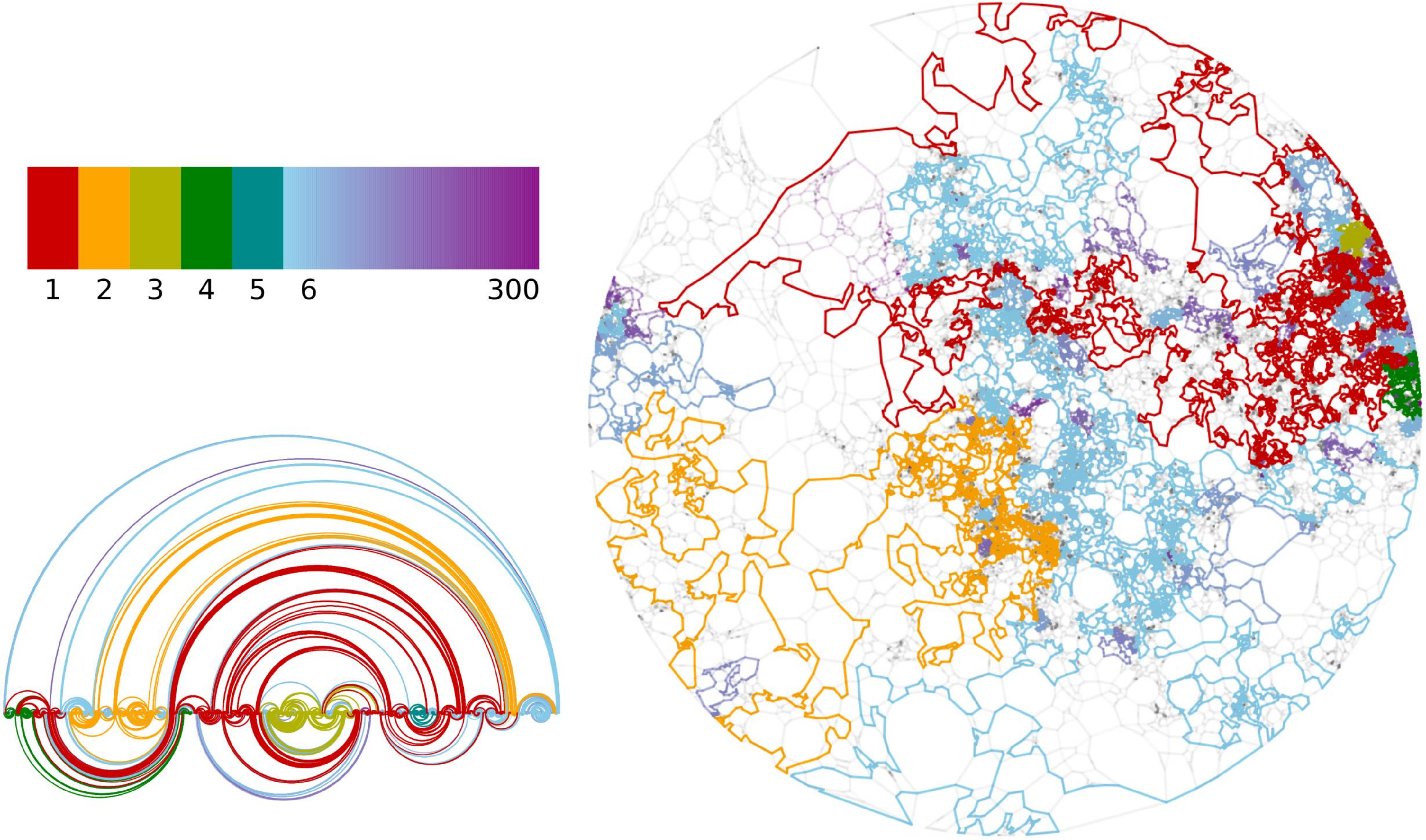


- GROMOV-HAUSDORFF topology for metric spaces
- Using some EMBEDDING

CONJECTURE: (B., Gwynne, Park, '22)

(M_n, P_n, Γ_n) converges under an appropriate scaling limit to a

$$\underbrace{\sqrt{2}\text{-LQG-measure} + \mathrm{SLE}_8}_{\text{Same as planar maps + spanning tree}} + \underbrace{\mathrm{CLE}_6}_{\text{Same as CRITICAL PERCOLATION}}$$


1 2 3 4 5 6 300

SOME QUESTIONS: (Loops are a complicated functional of the walks)

① How many loops?

- FÉRAY-THÉVENIN (2022, IMRN): $\#\text{loop} \sim c \cdot n$, where c is a complicated sum over meanders.

② What is the "SIZE" of the largest loop?

CONJECTURE (Borga-Gwynne-Park)

$$\#\text{ vertices of the } k\text{-th largest loop} \approx n^{\alpha + o(1)}, \quad \text{where } \alpha = \frac{3-\sqrt{2}}{2} \approx 0.7928$$

③ Does one loop dominate? Or, are there many large loops of similar "size"?

- Very related to the "infinite noodle" of Curien-Kozma-Sidoravicius-Tournier

↳ CONJ: There is NO INFINITE LOOP (confirmed + motivations)

④ What is the scaling limit as $n \to \infty$? CONJ from before

SEVERAL NUMERICAL SIMULATIONS (in our paper) CONFIRM the CONJECTURES.

Theorem: (B., Gwynne, Park, '22)

$3.55 \leq d \leq 3.63$

- Let d be the dimension of $\sqrt{2}$-LQG (just think of it as a constant)
- Let (M_n, P_n, Γ_n) be a uniform meandric system of size $n \in \mathbb{N}$. Then

$$\#\text{vertices of largest loop in } \Gamma_n \geq n^{1/d + o(1)} \geq n^{0.275}.$$

Proof:

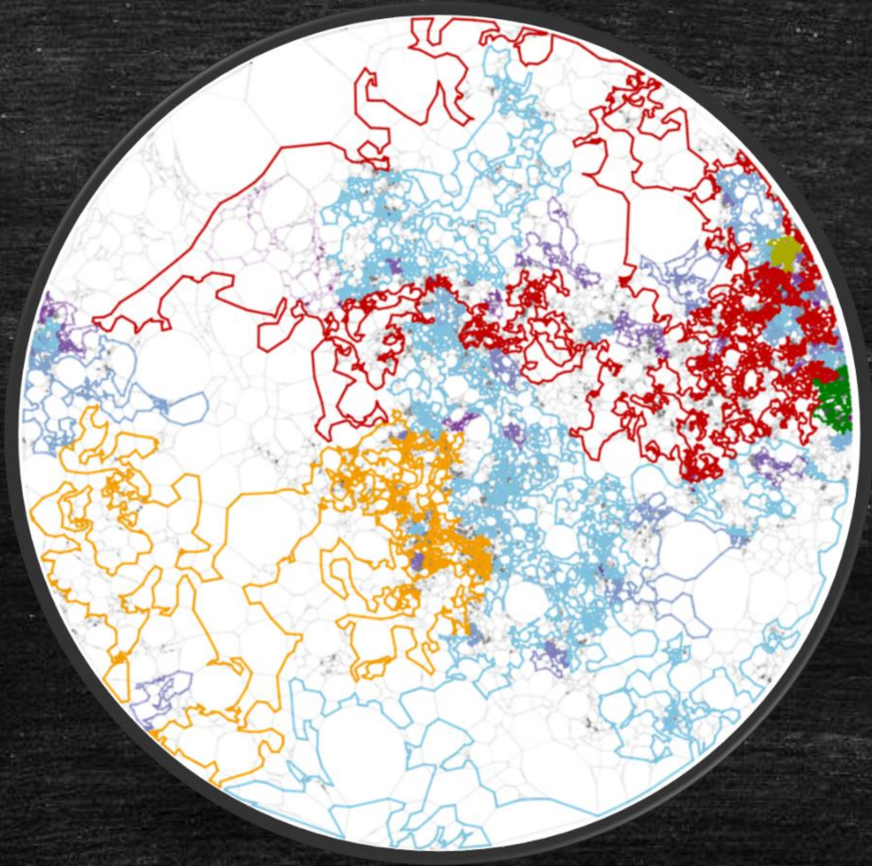
- STEP 1: Use a discrete parity argument to show that $\exists$ a large loop in Γ_n (w.r.t. the graph-metric induced by M_n).
- STEP 2: SLE/LQG arguments to lower-bound graph-distances
  - ↳ tools: MATING-OF-TREES (Miller–Sheffield) / LQG-METRIC (Gwynne, Miller/Ding, Dubedat, Dunlap, Falconet)

- Our theorem implies that there are (almost) macroscopic loops in Γ_n w.r.t. to M_n (which "surive in the scaling limit").
- Far from the conjecture ($\alpha \approx 0.793$) but better than previous results ($\log(n)$)

FINAL COMMENTS:

- We also proved that:
  - $\nexists$ infinite noodle in "half-plane meandric systems"
- Meandric systems are a fully-packed $O(\mathrm{o}\times 1)$-loop model, but we suggested an interpretation as percolation on planar maps ($\gamma^2 \neq 16/\kappa$)
- Both meandric systems & meanders are MISS-MATCHED MODELS and these models are very poorly understood compared with MATCHED-MODELS ($\gamma^2 = 16/\kappa$) (spanning-tree maps, percolation-maps-bipolar-orientations...) FK-maps

Thank you!



Thanks to all the people that contributed to the development of this area of Mathematics,
giving to us these beautiful random objects;
and sorry for the "magic" in my talk hiding lots of sophisticated ideas.

Jacopo Borga