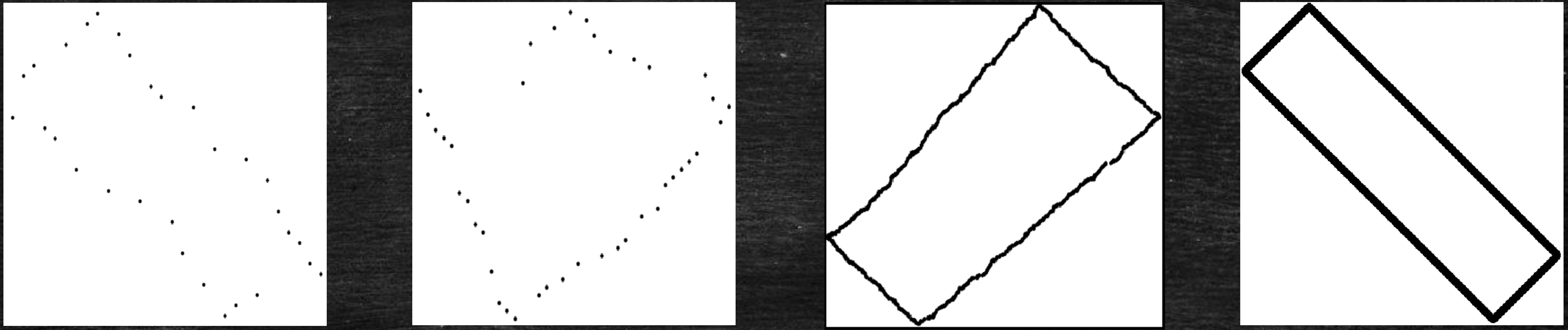


PHASE TRANSITION FOR ALMOST SQUARE PERMUTATIONS

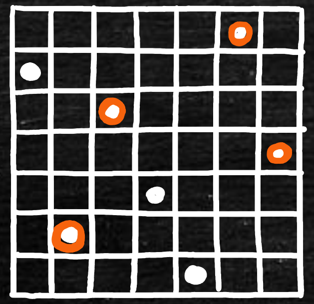
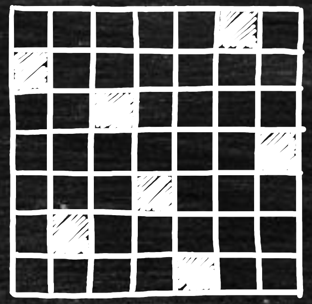
(joint work with E. Slivken & E. Duchi)

J. BORGA, UZH

A GRAPHICAL POINT OF VIEW ON PERMUTATIONS

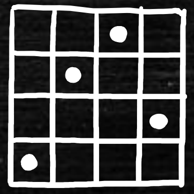
We look at permutations from a geometric perspective:

Consider the permutation $\sigma = 6\ 2\ 5\ 3\ 1\ 7\ 4$

$\sigma =$  $\rightsquigarrow$ $\mu_\sigma =$ 

Probability measure on the unit-square with uniform marginals

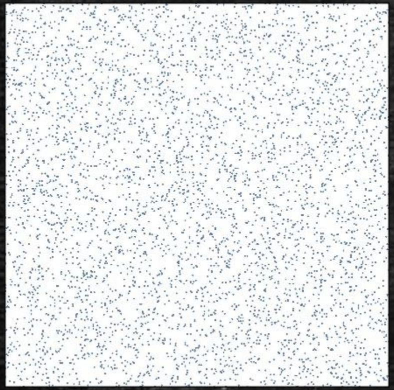
Def: An occurence of a pattern $\pi \in \mathcal{S}_k$ in $\sigma \in \mathcal{S}$ is a subsequence $\sigma(i_1)\dots\sigma(i_k) \in \mathcal{S}_k$ order-isomorphic to π.

Example: Occurrences of $\pi = 1342 \rightsquigarrow$ 

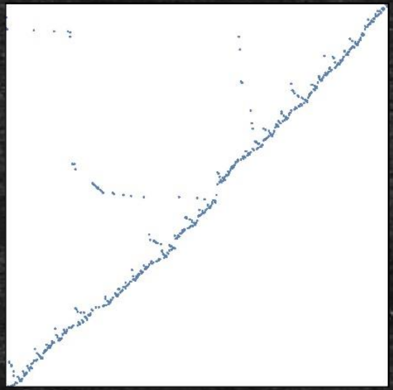
Winkler, Kenyon, Radin, Kral, Bevan, ...
Lebesgue measure

Hoffman, Rizzolo, Slivken
Brownian excursion

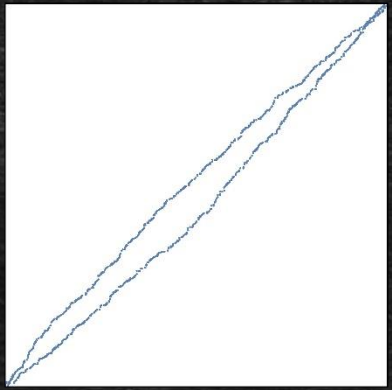
Traceless Dyson Brownian bridge



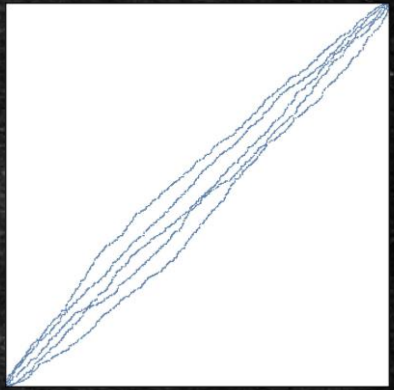
$\mathfrak{S}$



Av(231)



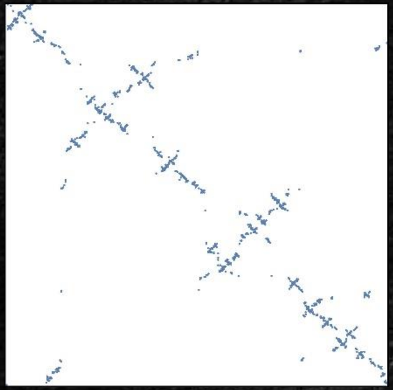
Av(321)



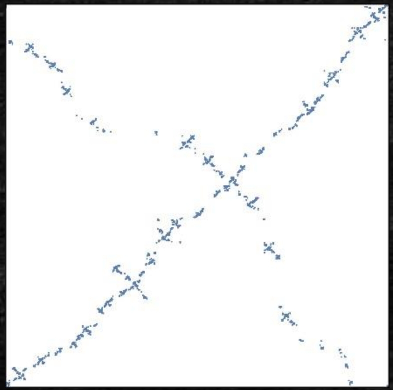
Av(654321)

Bassino, Bouvel, Féray, Gerin, Maazoun, Pierrot, B., Stufler
Continuum Random Tree

B. & Maazoun
flows of SDEs + LQG

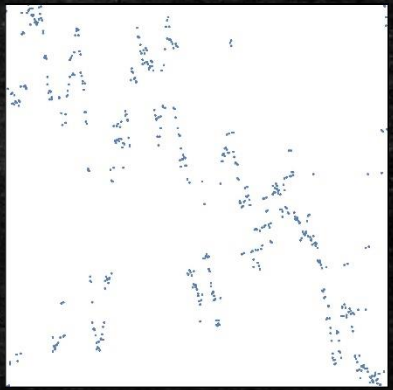


Av(2413, 3142)



SC(Av(321))

Baxter



Semi-Baxter

Def: A **PERMUTON** is a probability measure on the square $[0,1]^2$ with uniform marginals.

Remark: We have a natural notion of convergence of such objects: the WEAK CONVERGENCE. This defines a nice compact space.

$\Longrightarrow$ Limits of permutons are permutons, i.e., potential limits of sequences of permutations also have uniform marginals.

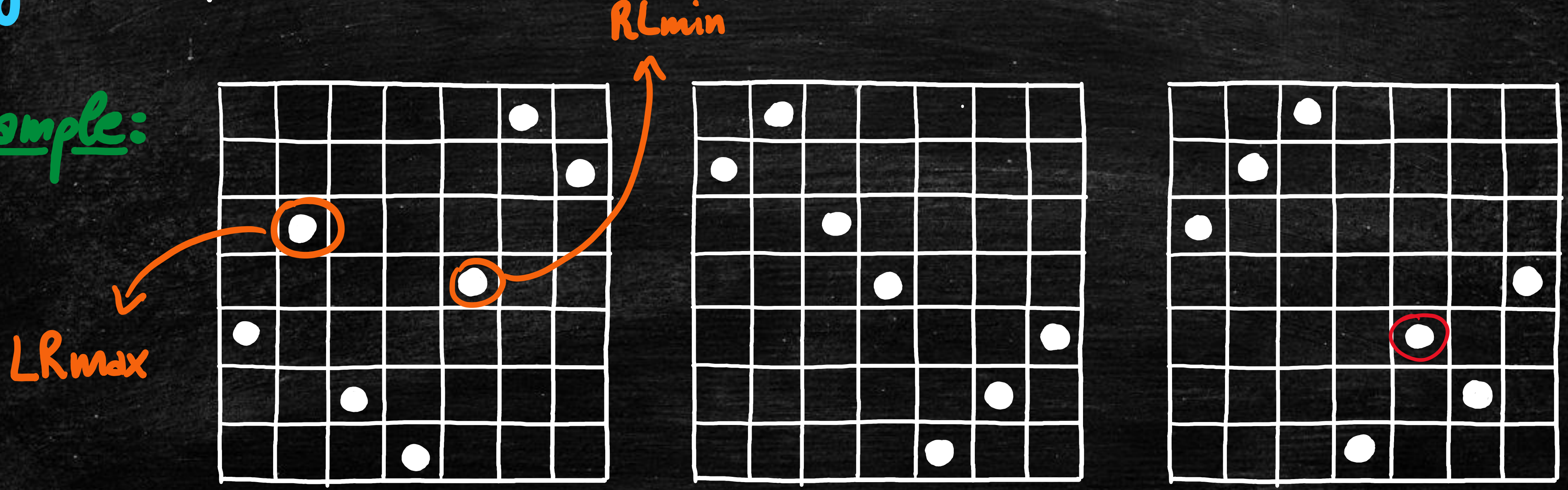
Moreover, every permuton is the limit of a sequence of μ_{σ_n}!

(Almost) square permutations

Def: A RECORD of a permutation is either a maximum or a minimum, from the left or from the right.

Def: If a point is a record, it is called EXTERNAL, otherwise INTERNAL

Example:



Def: Permutations with no internal points are called SQUARE PERMUTATIONS & permutations with a fixed number of internal points are called ALMOST SQUARE PERMUTATIONS.

Notation: $Asq(n,k)$ = set of permutations with n external points and k internal points.

$Sq(n) = Asq(n,0)$ = set of square perm. of size n.

ENUMERATION:

- $|Sq(n)| = 2(n+2)4^{n-3} - \underbrace{4(2n-5)\binom{2n-6}{n-3}}_{o(n4^{n-3})} \sim 2(n+2)4^{n-3}$

  Mansour & Severini (2007)
  Duchi & Poulalhon (2008)

  Disanto, Duchi, Rinaldi, Schaeffer. (2011)

- The gen. fct. for $(Asq(n,k))_{n\geq 0}$ is known only for $k = 1, 2, 3$.

GOALS OF THE PROJECT

- Asymptotic enumeration for $|Asq(n,k)|$;
- Investigate the permuton limit of (almost) square permutations;
- Investigate fluctuations for (almost) square perm;
- Show that our approach is quite general and it works also for other families of permutations;
- BONUS: The terminology "SQUARE PERMUTATIONS" and "ALMOST SQUARE PERMUTATIONS" is a very poor choice ☹

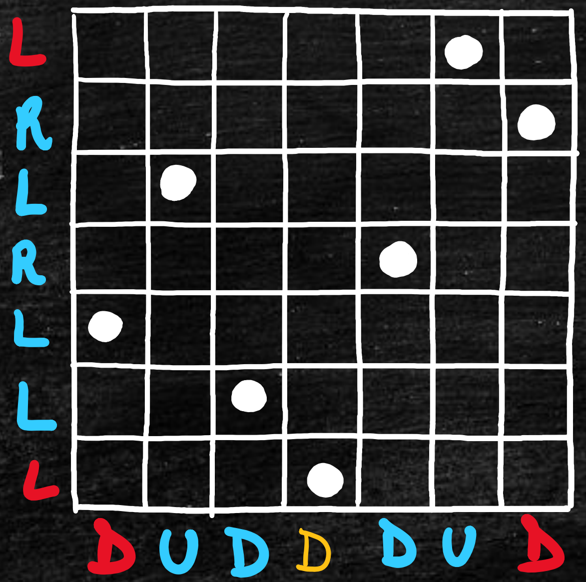
Sampling uniform square permutations

We consider the following projection map:

$$\varphi : Sq(n) \longrightarrow \{U,D\}^n \times \{L,R\}^n \times [n]$$

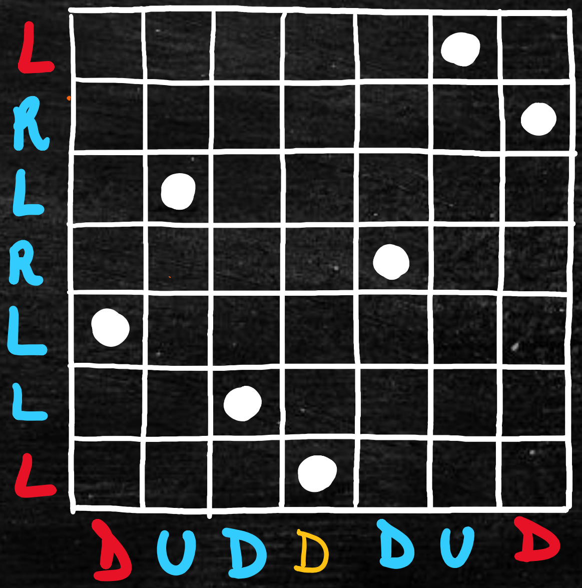
The space of anchored pairs of sequences of labels.

Y



$\longmapsto (X, Y, z_0)$

X $\hookrightarrow z_0 = 4$



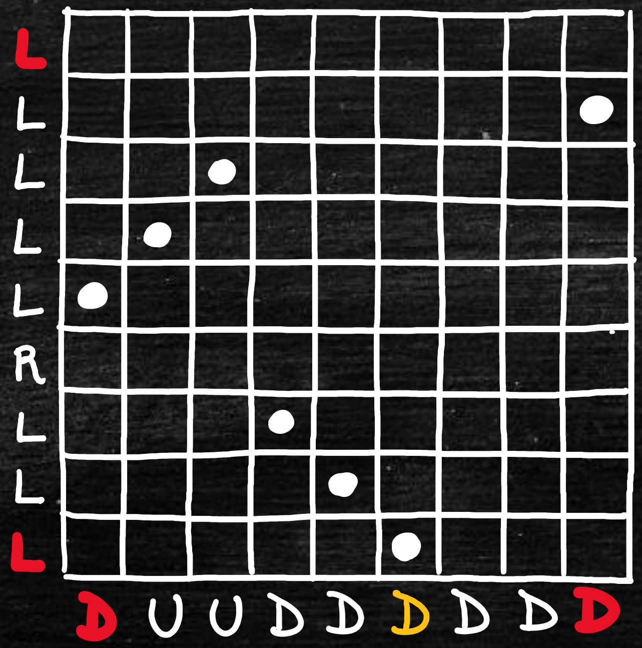
Def: We say that (X, Y, z_0) is a good anchored pair of sequences if

$$X_1 = X_n = X_{z_0} = D \quad \text{and} \quad Y_1 = Y_n = L.$$

Note that # of good anchored pairs $= 2(n+2)\,4^{n-3}$.

PROBLEM: We cannot reconstruct a square permutation from every good anchored pairs of sequences.

Counterexample:



← FAIL ☹

Why?

TOO MANY Ls & Ds!

We need some regularity conditions on our sequences that are satisfied by "asymptotically almost all" good anchored pairs.

Notation:
- $ct_D(i)$ = # of Ds in X up to (and including) position i
- $pos_D(i)$ = index of the i-th D in X.

PETROV CONDITIONS:

(1) $\left|ct_D(i) - ct_D(j) - \frac{1}{2}(i-j)\right| < n^{.4}$, for $|i-j| < n^{.6}$;

(2) $\left|ct_D(i) - ct_D(j) - \frac{1}{2}(i-j)\right| < \frac{1}{2}|i-j|^{.6}$, for $|i-j| > n^{.3}$;

(3) $\left|pos_D(i) - pos_D(j) - 2(i-j)\right| < n^{.4}$, for $|i-j| < n^{.6}$;

(4) $\left|pos_D(i) - pos_D(j) - 2(i-j)\right| < 2|i-j|^{.6}$, for $|i-j| > n^{.3}$.

Def: We say that a good anchored pair (X, Y, z_0) is *regular* if:

- X and Y satisfy the Petrov conditions,
- $n^q \leq z_0 \leq n - n^q$

We denote by Ω_n the space of regular anchored pairs of size n.

Lemma 1: $\varphi^{-1} : \Omega_n \longrightarrow Sq(n)$ is well-defined and injective.

Lemma 2: Let (X, Y, z_0) be chosen independently and uniformly at random from $\{U, D\}^n \times \{L, R\}^n \times [n]$, then $\mathbb{P}\left((X, Y, z_0) \in \Omega_n \,\middle|\, \begin{matrix}(X, Y, z_0) \\ \text{is good}\end{matrix} \right) \geq 1 - o(1)$.

Theorem: With probability $1 - o(1)$ a uniform square permutation σ_n of size n belongs to $\varphi^{-1}(\Omega_n)$.

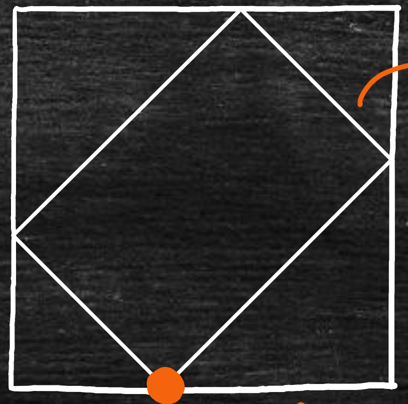
Proof:

$$\mathbb{P}\left(\sigma_n \in \varphi^{-1}(\Omega_n)\right) \underset{\varphi^{-1}\text{ is injective}}{=} \frac{|\Omega_n|}{|Sq(n)|} = \frac{\overbrace{2(n+2)4^{n-3}}^{\#\text{ good anchored pairs}}\ \overbrace{(1-o(1))}^{\text{Lemma 2}}}{\underbrace{2(n+2)4^{n-3}(1-o(1))}_{\text{"enumerative result"}}} \longrightarrow 1. \quad \square$$

THEOREM:

Let σ_n be a uniform random square permutation of size n, then

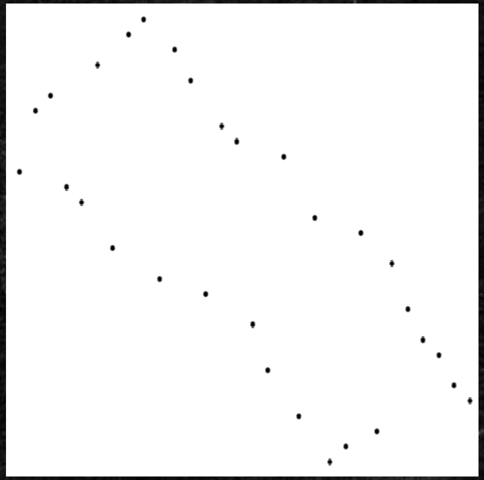
$$\mu_{\sigma_n} \xrightarrow{d} \mu^z =$$



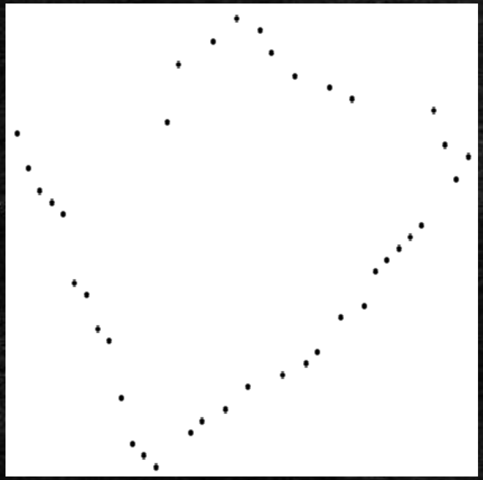
Lebesgue measure on the rectangle with total mass 1.

$z \sim \mathrm{Unif}([0,1])$

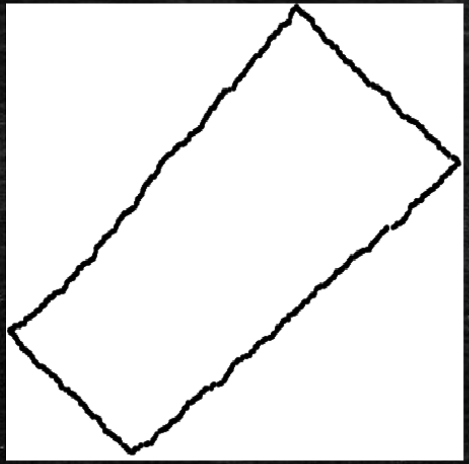
Simulations:



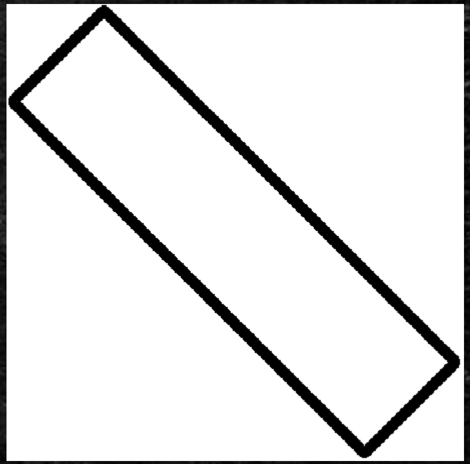
$n=30$



$n=40$

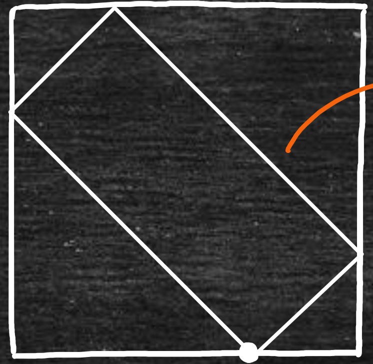


$n=1000$



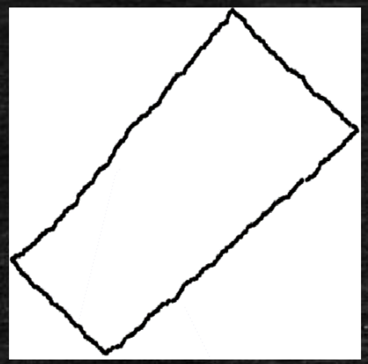
$n=1000000$

Proof: For every $z \in (0,1)$, we define the permuton

$\mu^z =$ 

Lebesgue measure on the rectangle with total mass 1

Then, for every square permutation $\sigma_n \in \varphi^{-1}(\Omega_n)$, we consider the

permuton $\mu_{\sigma_n} =$ 

$\mu^{z_n} =$ 

We show that

$$\sup_{\sigma^n \in \varphi^{-1}(\Omega_n)} d_\square(\mu_{\sigma_n}, \mu^{z_n}) < C n^{-.4}; \quad \text{where } z_n = \sigma_n^{-1}(1)/n.$$

↳ metric for the permuton topology.

What about "adding" internal points?

Notation: $Asq(n,k)$ = Permutations of size $n+k$ with k internal points

Theorem: Let $k = o(\sqrt{n})$, then

$$|Asq(n,k)| \underset{n\to\infty}{\sim} \frac{k!\,2^{k+1}\,n^{2k+1}\,4^{n-3}}{(2k+1)!} \sim \frac{k!\,2^{k}\,n^{2k}}{(2k+1)!}\,|Sq(n)|$$

Theorem: Let $k = o(n)$, then

$$\log(|Asq(n,k)|) = \log\left(\frac{k!\,2^{k+1}\,n^{2k+1}\,4^{n-3}}{(2k+1)!}\right) + O(k).$$

IDEA OF THE PROOF:

permutations with k internal points that "can be constructed" from σ

$$|Asq(n,k)| = \sum_{\sigma \in Sq(n)} |Asq(\sigma,k)| = \sum_{\sigma \in Sq(n)} \frac{1}{k!} |\mathcal{I}(\sigma,k)|$$

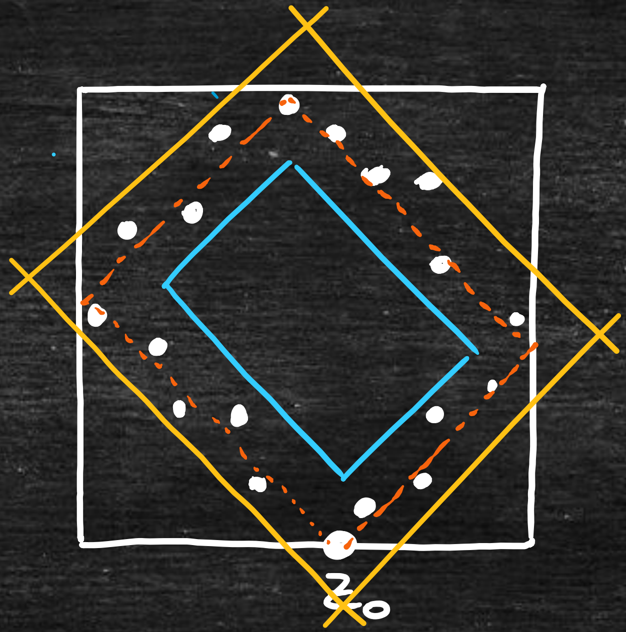
Set of sequences $((x_1,y_1),\dots,(x_k,y_k))$ s.t. the insertion of the points (x_i,y_i) $\forall i$ in σ leads to a permutation in $Asq(\sigma,k)$

So we need bounds for $|\mathcal{I}(\sigma,1)|$ that can then be iterated to obtain bounds for $|\mathcal{I}(\sigma,k)|$.

$\sigma \in Sq(n)$

$\varphi(\sigma)$ is regular

$z_0 = \sigma^{-1}(1)$



Lemma: Let $\sigma \in Sq(n)$ be s.t. $\varphi(\sigma)$ is regular. Then

$$2(z_0 - cn^{\epsilon})(n - z_0 - cn^{\epsilon}) \le |\mathcal{I}(\sigma,1)| \le 2(z_0 + cn^{\epsilon})(n - z_0 + cn^{\epsilon})$$

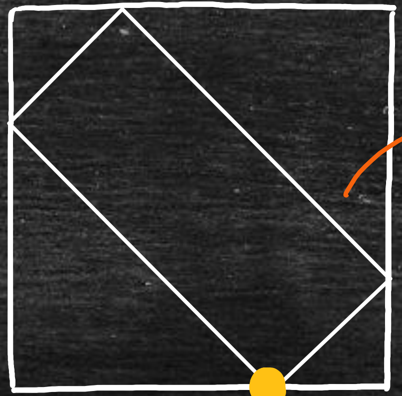
where $z_0 = \sigma^{-1}(1)$ and c is a positive const. indep. of σ.

$$|Asq(n,k)| = \sum_{\sigma \in Sq(n)} \frac{1}{k!} \underbrace{|\mathcal{I}(\sigma,k)|}_{\sim |\mathcal{I}(\sigma,1)|^k} \sim |Sq(n)| \cdot \frac{1}{k!} 2^k n^{2k} \underbrace{\int_0^1 (t(1-t))^k \, dt}_{= (k!)^2/(2k+1)!}$$

Theorem: Fix $\kappa > 0$. Let σ_n be a uniform permutation in $Asq(n,\kappa)$.

Then

$$\mu_{\sigma_n} \xrightarrow{d} \mu^{z_\kappa} =$$



z_κ

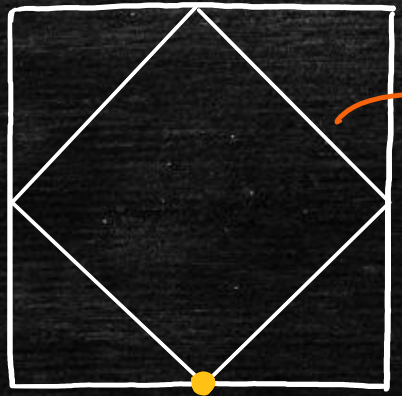
→ Lebesgue measure on the rectangle with total mass 1

where

$$\mathbb{P}(z_\kappa < s) = (2\kappa+1)\binom{2\kappa}{\kappa}\int_0^s (t(1-t))^\kappa \, dt \quad \forall s \in (0,1).$$

Moreover, if $\kappa \to +\infty$ and $\kappa = o(n)$ then

$$\mu_{\sigma_n} \xrightarrow{d} \mu^{1/2} =$$

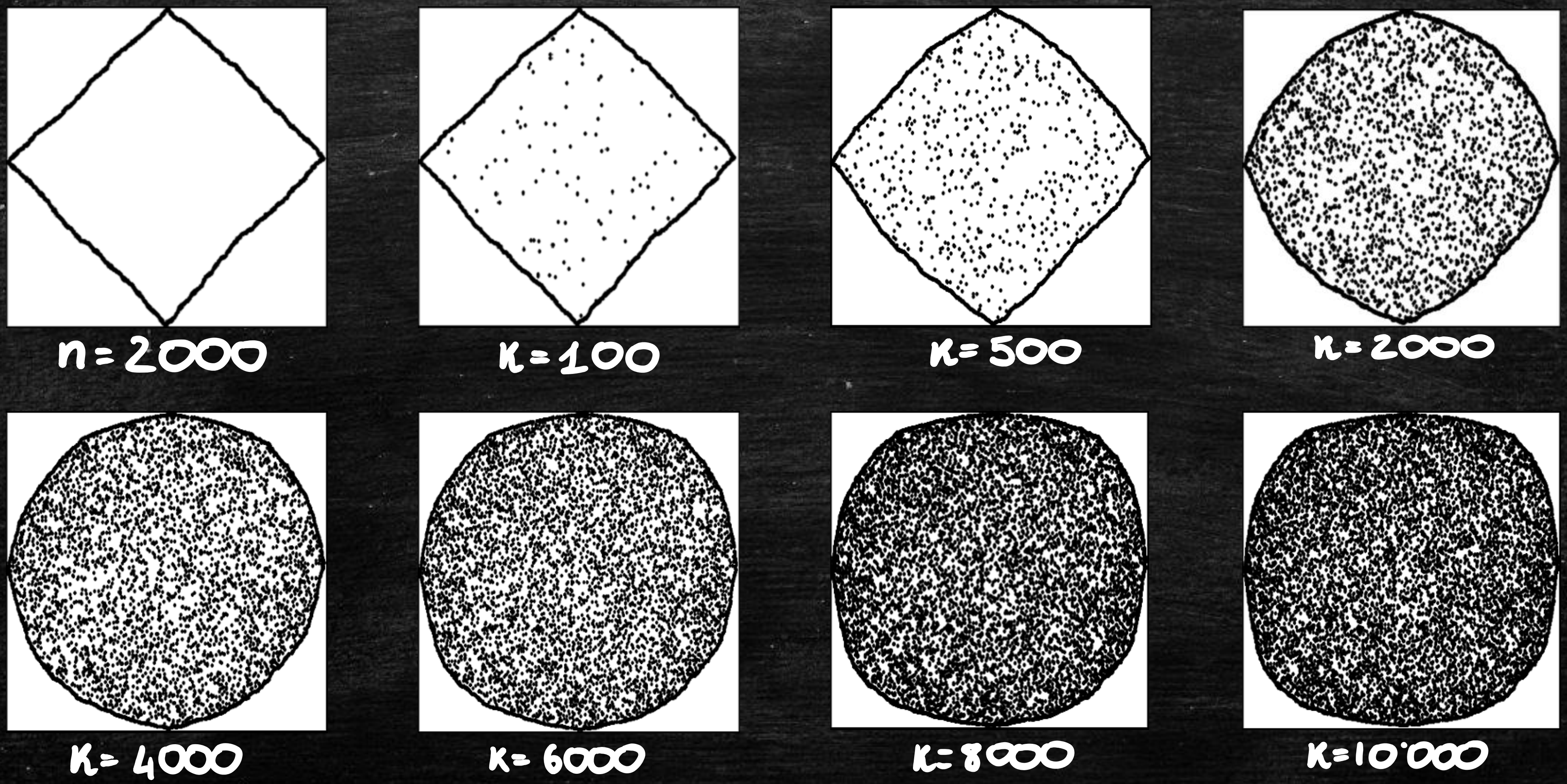


$1/2$

→ Lebesgue measure on the square with total mass 1

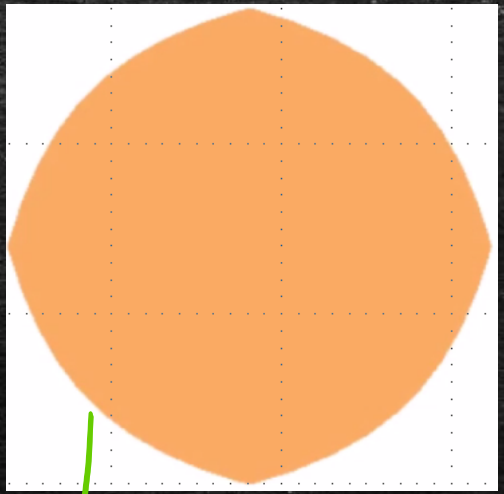
MORAL: Square permutations are typically rectangular & almost square permutations are typically square

Next step ?!



CONJECTURE: Let k, n s.t. $\frac{k}{n} \longrightarrow \ell \in (0,\infty]$. Let σ^n be uniform in $\mathrm{Asq}(n,k)$. If $\ell < \infty$ then

$$\mu_{\sigma^n} \xrightarrow{d}$$



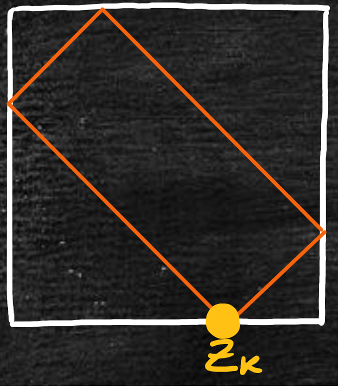
with
- Leb. measure on the boundary with total mass $\frac{1}{\ell+1}$
- Leb. measure on the interior with total mass $\ell/_{\ell+1}$

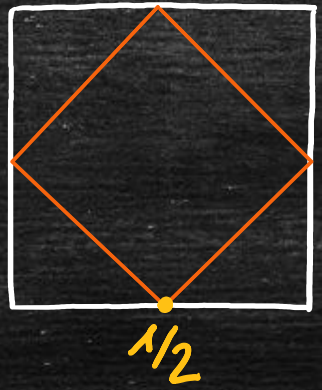
$$\left(\frac{e^{A(1-t)}-1}{2(e^A-1)}, \frac{e^{At}-1}{2(e^A-1)} \right)_{t\in[0,1]}$$

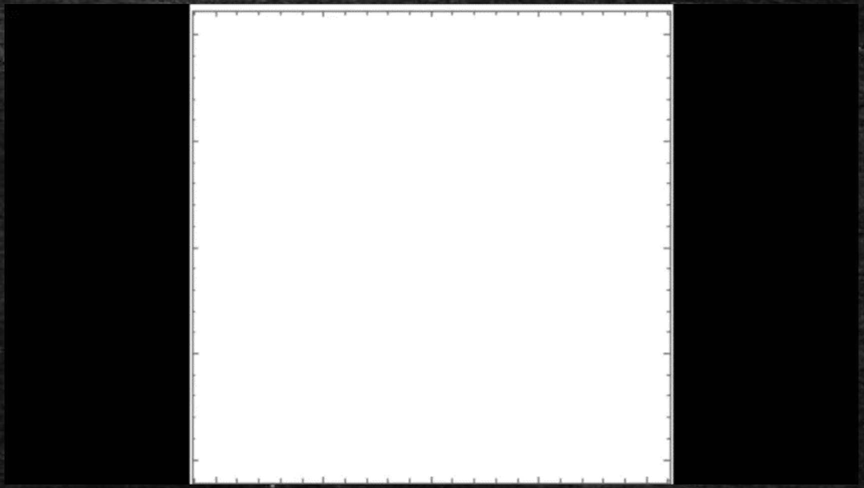
where $\ell = \frac{e^A-1-A}{A}$.

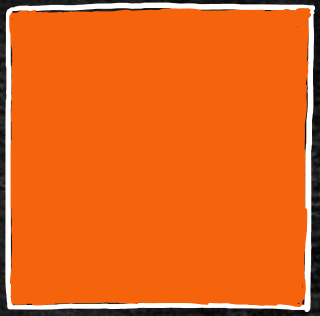
If $\ell = +\infty$, then $\mu_{\sigma^n} \longrightarrow \mathrm{Leb}([0,1]^2)$.

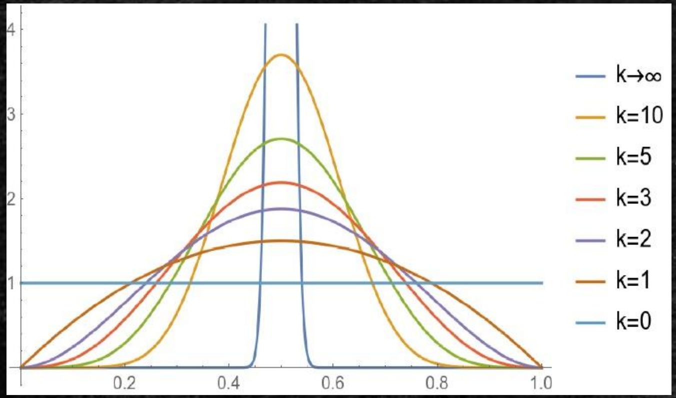
THE PHASE TRANSITION









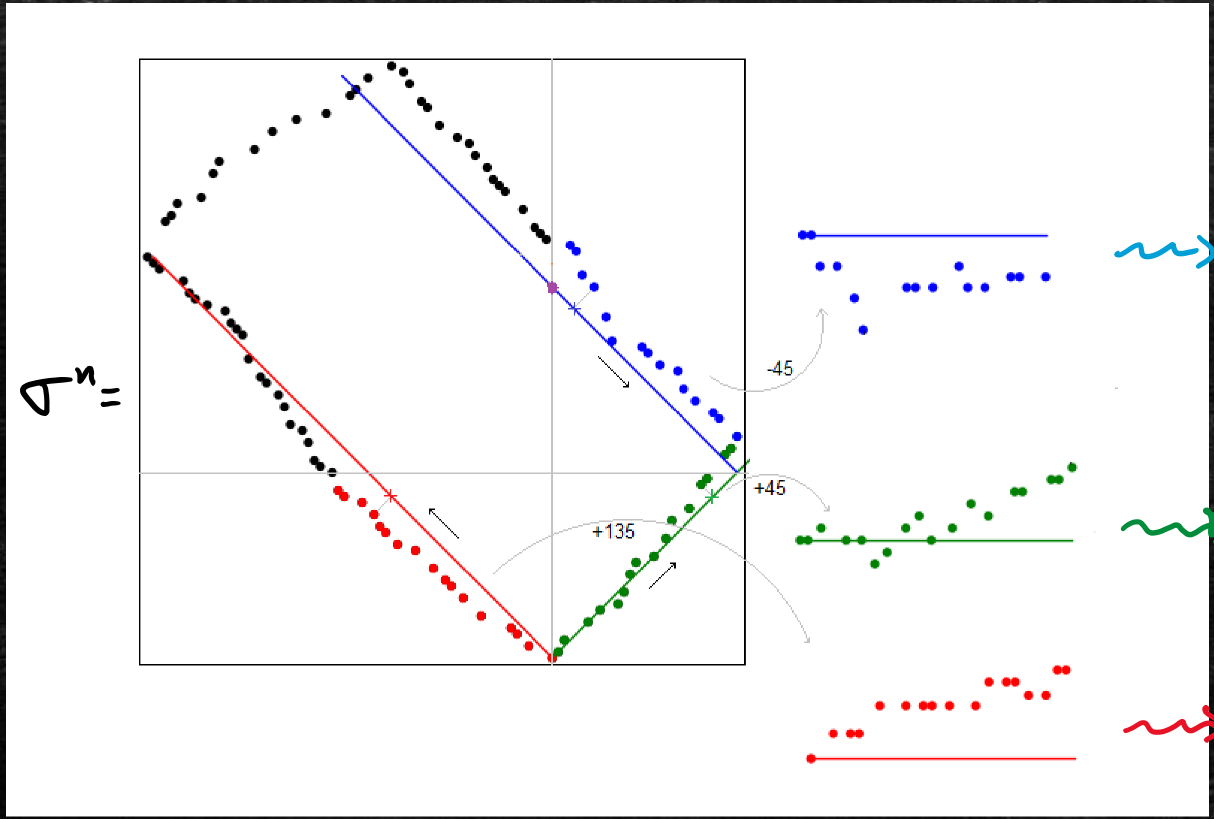


0 $\quad k$ fixed $\quad k \to \infty,\ k = o(n) \quad \frac{k}{n} \to \ell \in (0,\infty) \quad \frac{k}{n} \to +\infty$

Fluctuations

QUESTION: What happens if instead of a factor n we rescale distances by a factor $\sqrt{n}$?

$\sigma^n =$



$\rightsquigarrow \left(F^{\sigma^n}(t)\right)_{t\in[0,1]}$

$\rightsquigarrow \left(F^{\sigma^n}(t)\right)_{t\in[0,1]}$

$\rightsquigarrow \left(F^{\sigma^n}(t)\right)_{t\in[0,1]}$

THEOREM: Let σ^n be a uniform random square permutation of size n, and $B_1(t), B_2(t), B_3(t), B_4(t)$ be four independent standard Brownian motions on $[0,1]$.

Conditioning on $z_0 = t_n$, with $\frac{n}{2} + C n^{.6} < t_n \leq n - n^{.9}$, we have

$$\left(F^{\sigma^n}(t), F^{\sigma^n}(t), F^{\sigma^n}(t)\right)_{t\in[0,1]} \xrightarrow{d} \left(B_1(t)+B_2(t), B_3(t)-B_1(t), B_4(t)-B_2(t)\right)_{t\in[0,1]}.$$

The case of 321-avoiders

Def: 321-avoiding permutations are permutations s.t. the longest decreasing subsequence has size at most 2.

Fact: 321-avoiding permutations can be partitioned into two increasing subsequences, one weakly above the diagonal and one strictly below the diagonal.

Notation: $\mathrm{Asq}(\mathrm{Av}_n(321), k)$ = Set of permutations with k internal points and n external points avoiding 321

$$\cap\!\!|$$

$$\mathrm{Asq}(n,k)$$

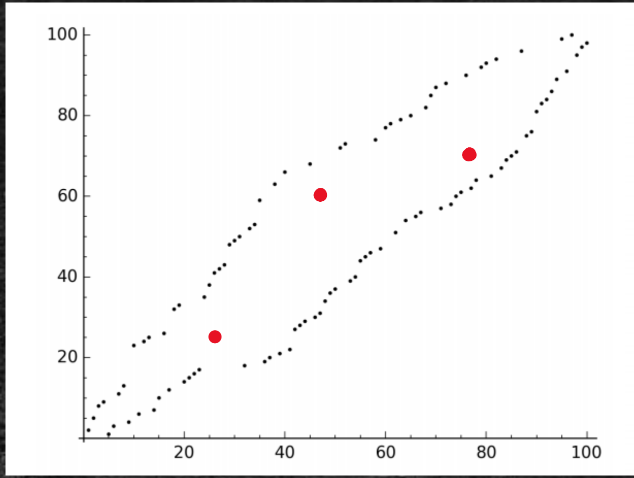
Theorem: Fix $\kappa > 0$. Then as $n \to \infty$,

$$\left|\mathrm{Asq}(\mathrm{Av}_n(321), \kappa)\right| \sim \frac{(2n)^{3\kappa/2}}{\kappa!} \cdot \underbrace{c_n}_{\text{n-th Catalan number}} \cdot \mathbb{E}\Big[\Big(\int_0^1 \overbrace{e_t}^{\text{Brownian excursion}}\, dt\Big)^{\kappa}\Big]$$

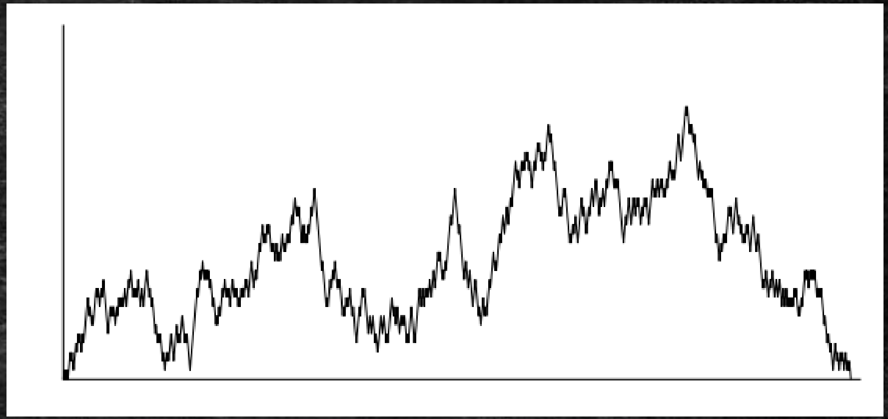
KEY INGREDIENTS:

- Probabilistic approach developed for (almost) square permutations
- Hoffman, Rizzolo, Slivken, 2015: the two increasing subsequences of a uniform 321-avoiding permutation converge to two Brownian excursions.

$\tau_n^\kappa =$



$\rightsquigarrow F_{\tau_n^\kappa}(t) =$



Def: Fix $\kappa>0$. The κ-biased Brownian excursion $(e_t^\kappa)_{t\in[0,1]}$ is a random continuos function with law

$$E\left[G(e_t^\kappa)\right] = E\left[\left(\int_0^1 e_t\,dt\right)^\kappa\right]^{-1} \cdot E\left[G(e_t)\left(\int_0^1 e_t\,dt\right)^\kappa\right] \quad \forall\, G \text{ continuous and bounded}$$

Theorem: Fix $\kappa>0$. Let τ_n^κ be a uniform perm. in $Asq(Av_n(321),\kappa)$, then

$$\left(F_{\tau_n^\kappa}(t)\right)_{t\in[0,1]} \xrightarrow{d} \left(e_t^\kappa\right)_{t\in[0,1]}$$

THANK YOU!