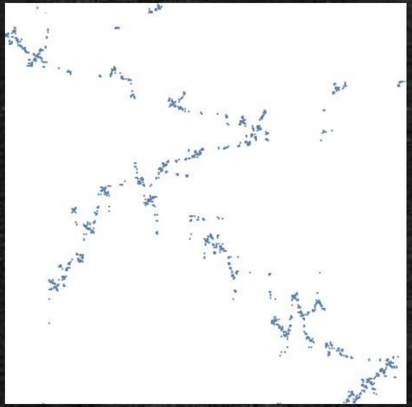
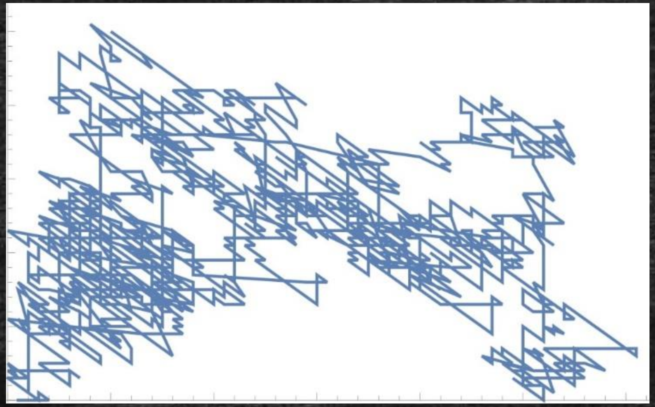
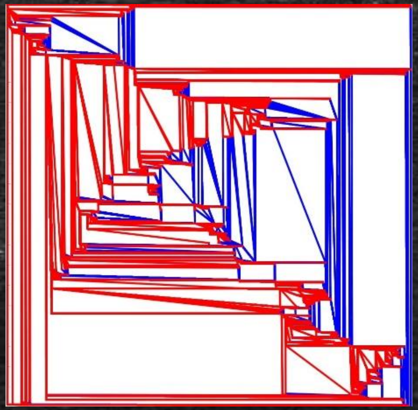
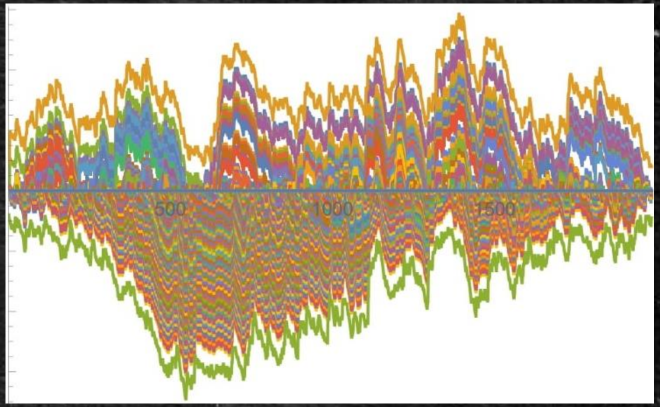








Scaling and local limits of Baxter permutations and bipolar orientations through coalescent-walk processes

Jacopo Borga

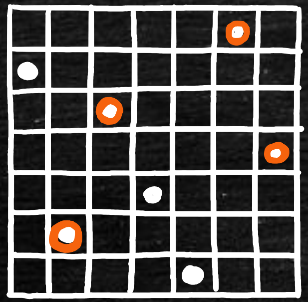
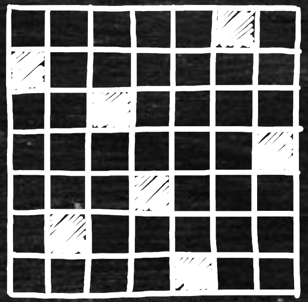
UZH ZÜRICH

(joint work with M. Maazoun)

Motivations(PART 1): Permuton limits

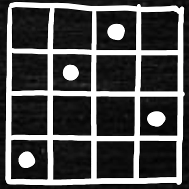
We look at permutations from a <u>geometric</u> <u>perspective</u>:

Consider the permutation $\sigma = 6\ 2\ 5\ 3\ 1\ 7\ 4$

$\sigma =$  $\rightsquigarrow$ $\mu_\sigma =$ 

Probability measure on the unit-square with uniform marginals

<u>Def</u>: An <u>OCCURENCE</u> of a pattern $\pi \in S_k$ in $\sigma \in S$ is a subsequence $\sigma(i_1)\dots\sigma(i_k) \in S_k$ order-isomorphic to π.

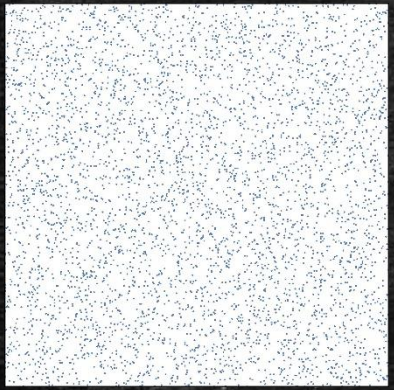
<u>Example</u>: Occurrences of $\pi = 1342 \rightsquigarrow$ 

Hoffman, Rizzolo, Slivken

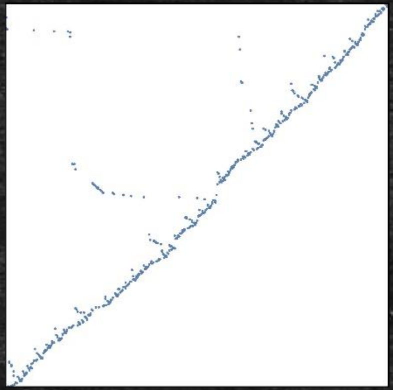
Lebesgue measure

Brownian excursion

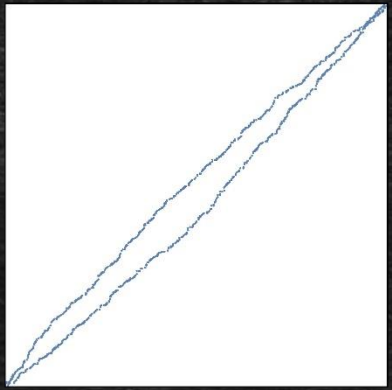
Traceless Dyson Brownian bridge



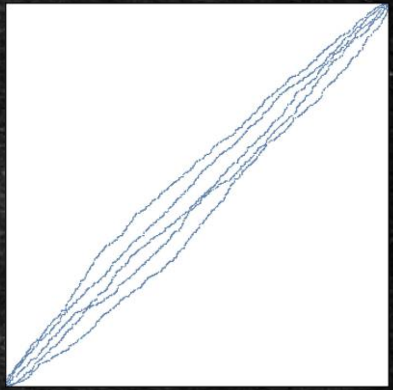
$\mathfrak{S}$



Av(231)



Av(321)

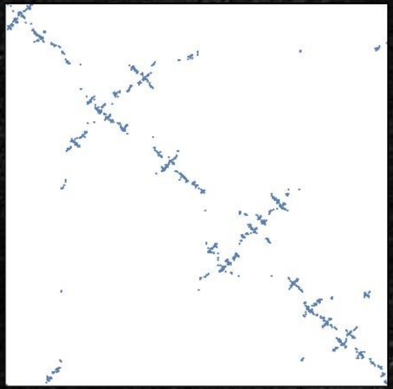


Av(654321)

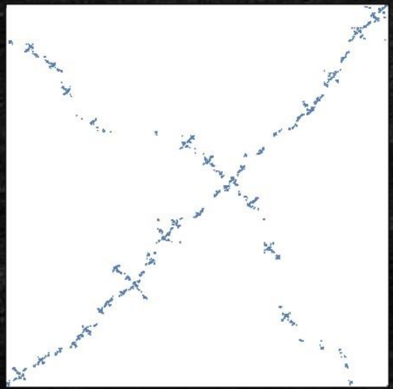
Bassino, Bouvel, Féray, Gerin, Maazoun, Pierrot, B., Stufler

Continuum Random Tree

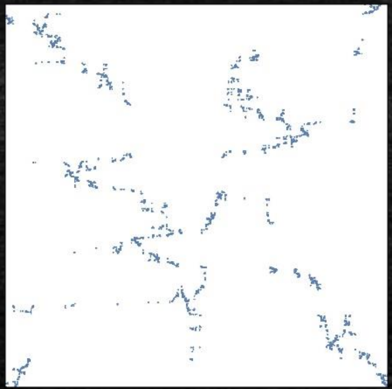
flows of SDEs + LQG



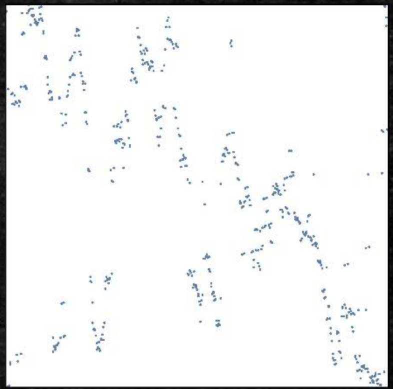
Av(2413, 3142)



$\mathcal{SC}$(Av(321))



Baxter



Semi-Baxter

Def: A **PERMUTON** is a probability measure on the square $[0,1]^2$ with uniform marginals.

Remark: We have a natural notion of convergence of such objects: the **WEAK CONVERGENCE**. This defines a nice compact Polish space.

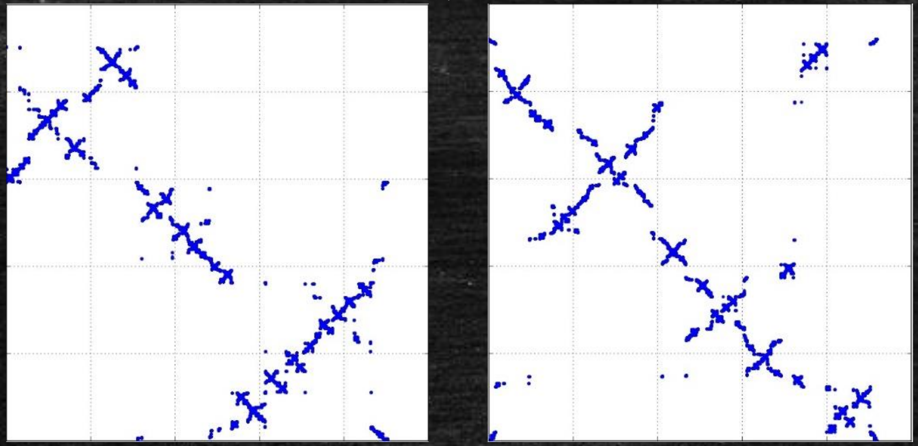
$\Longrightarrow$ limits of permutons are permutons, i.e., potential limits of sequences of permutations also have uniform marginals.

THEOREM [① Bassino, Bouvel, Féray, Gerin, Maazoun and Pierrot, 2016
② B., Bouvel, Féray, Stufler, 2019]

Let $\mathcal{C}$ be a substitution closed class.

Let σ^n be a uniform random permutation of size n in $\mathcal{C}$. Then

$$\mu_{\sigma^n} \xrightarrow{d} \mu_p =$$



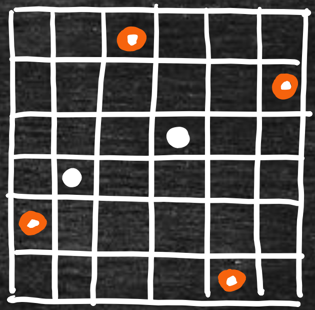
parameter $p \in [0,1]$ depending on $\mathcal{C}$

RANDOM BROWNIAN SEPARABLE PERMUTON

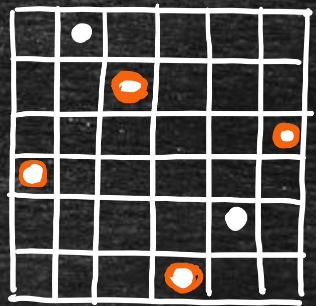
Example: $\mathcal{C} = Av(2413-3142)$ Separable permutations

Def: Baxter permutations are permutations avoiding the patterns

2413 & 3142.



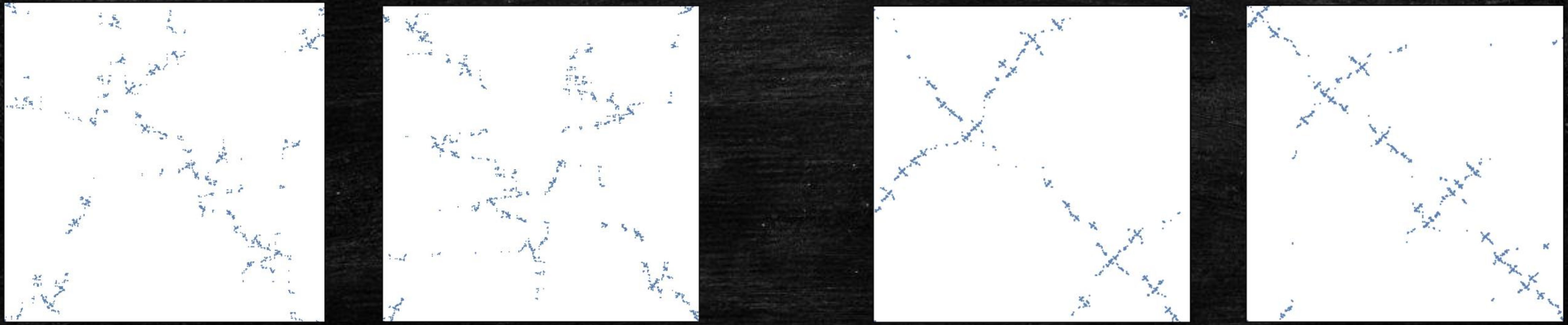
BAXTER



NOT BAXTER

Dokos & Pak (2014) explored the expected shape of doubly alternating Baxter permutations, i.e. Baxter perm. σ s.t. σ and σ^{-1} are alternating and they claimed that

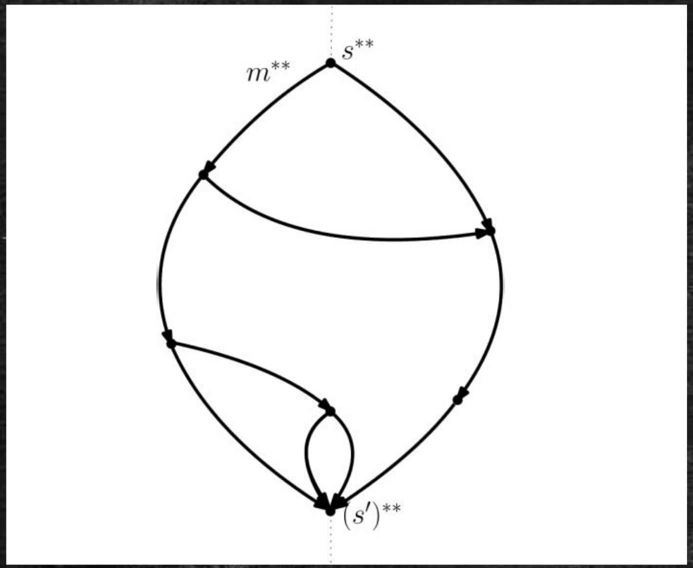
"IT WOULD BE NICE TO COMPUTE THE LIMIT SHAPE OF BAXTER PERMUTATIONS"

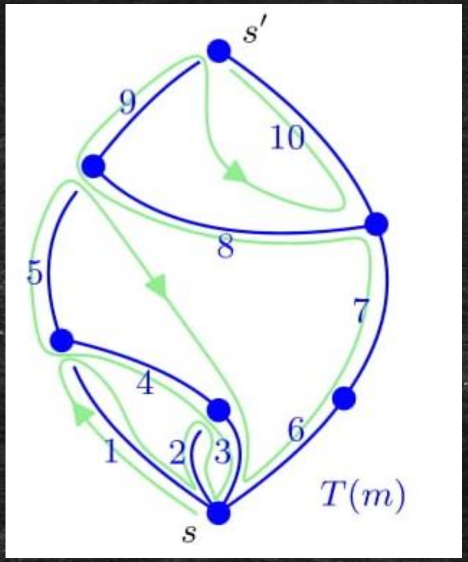


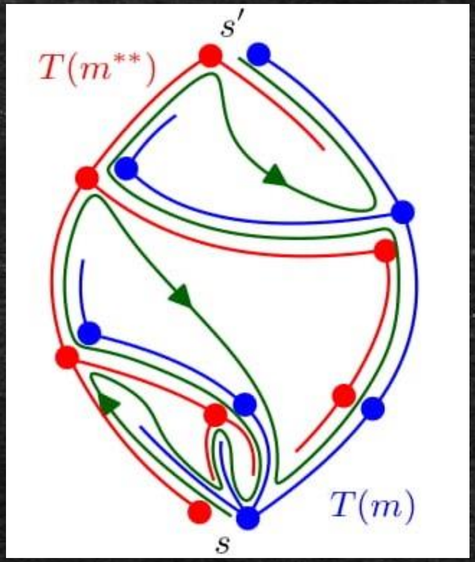
Motivations(PART 2): Bipolar orientations and walks in cones

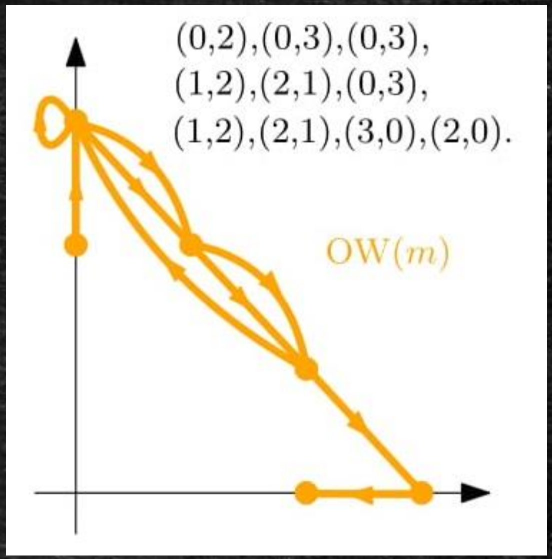
Bonichon, Bousquet-Mélou & Fusy (2011) showed that Baxter permutations are in bijection with plane bipolar orientations.

Def: A PLANE BIPOLAR ORIENTATION is a planar map (connected graphs properly embedded in the plane up to continuous deformations) equipped with an acyclic orientation of the edges with exactly one source (a vertex with only outgoing edges) and one sink (a vertex with only incoming edges) both on the outer face.







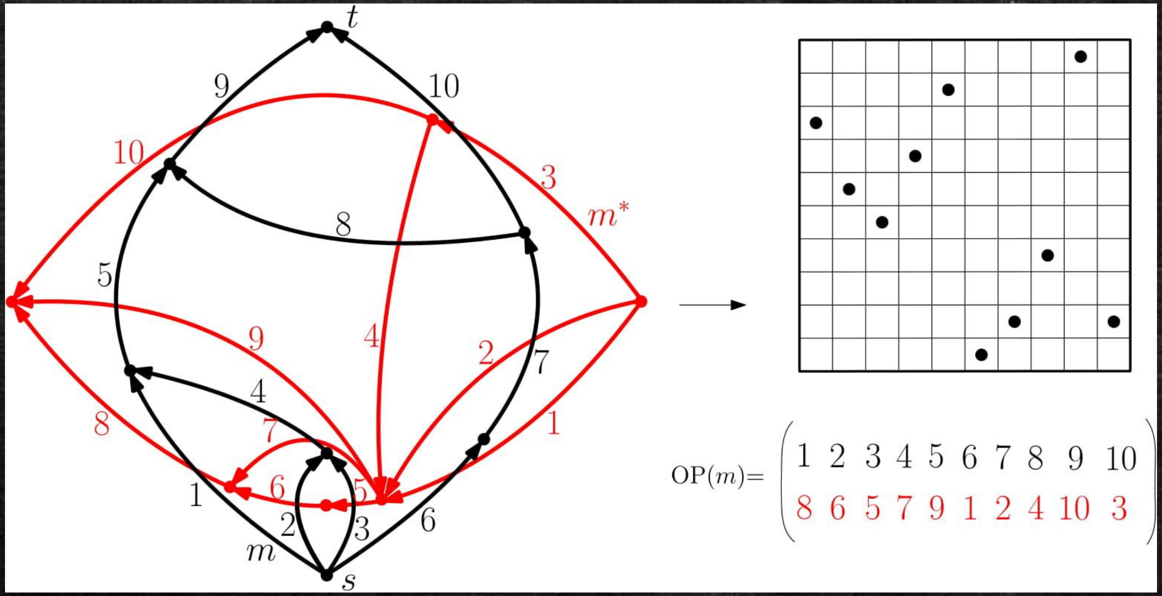


Kenyon, Miller, Sheffield & Wilson (2015) constructed the following bijection.

Def: Let $n \geq 1$ and m be a bipolar orientation with n edges. We define $OW(m) = (X_t, Y_t)_{1 \leq t \leq n} \in (\mathbb{Z}^2_{\geq 0})^n$ as follows: for $1 \leq t \leq n$, X_t is the height in the tree $T(m)$ of the bottom vertex of e_t and Y_t is the height in the tree $T(m^{**})$ of the top vertex of e_t.

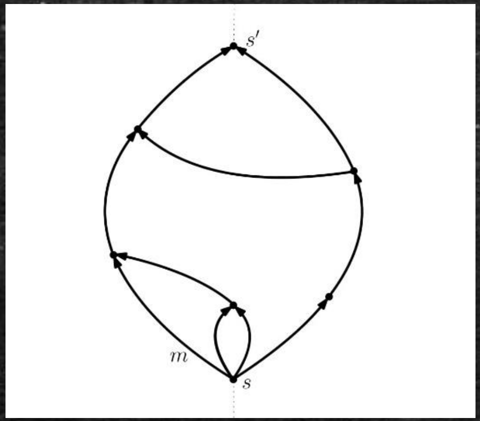
THEOREM: (Gwynne, Holden, Sun 2016)
The pairs of height functions for an infinite-volume random bipolar triangulation and its dual converge jointly in law to the two Brownian motions which encode the same $\sqrt{4/3}$-LQG surface decorated by both an SLE_{12} and the "dual" SLE_{12} which travels in a perpendicular direction.

Def: A **TANDEM WALK** is a two-dimensional walk in $\mathbb{Z}^2_{\geq 0}$ starting at $(0,h)$ and ending at $(k,0)$ with steps in $\{(+1,-1)\} \cup \{(-i,j) : i,j \geq 0\}$.

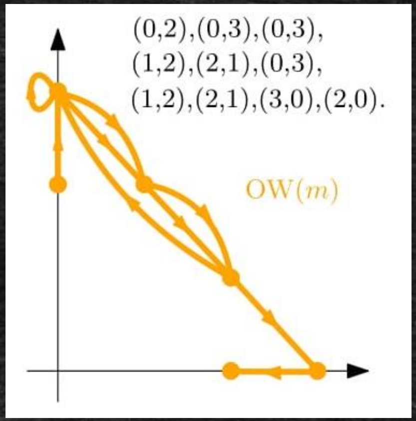


Def: Let $n \geq 1$ and m a bipolar orientation with n edges.
Let $OP(m)$ be the only permutation π such that for every $1 \leq i \leq n$, the i-th edge to be visited in the expolarion of $T(m)$ corresponds to the $\pi(i)$-th edge to be visited in the exploration of $T(m^*)$.

So far ...

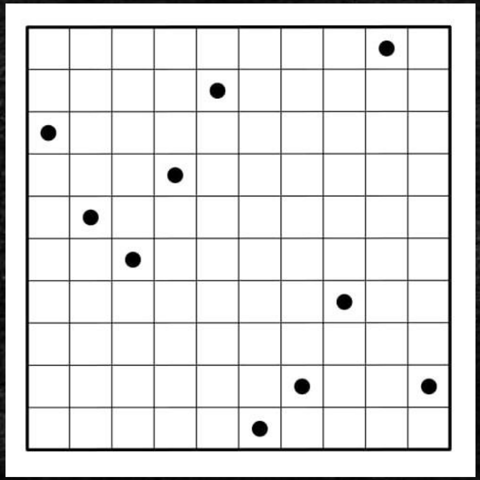


$\xrightarrow{OW}$



$\downarrow OP$

$OP \circ OW^{-1}$



WE WANT "TO READ" THE PATTERNS OF A PERMUTATION IN THE CORRESPONDING WALK

Coalescent-walk processes

Let $W_t = (X_t, Y_t)$ be a tandem walk & $\sigma = OP \circ OW^{-1}(W)$ be the corresponding Baxter permutation.

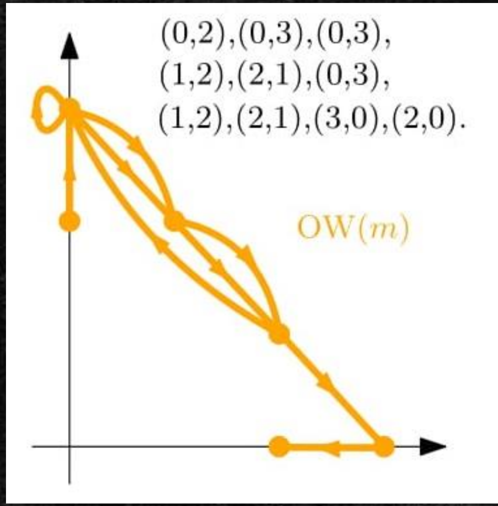
IDEA: Given $i<j$, we want to find a way in order to "read" in W_t if $\sigma(i)<\sigma(j)$ or $\sigma(j)<\sigma(i)$.

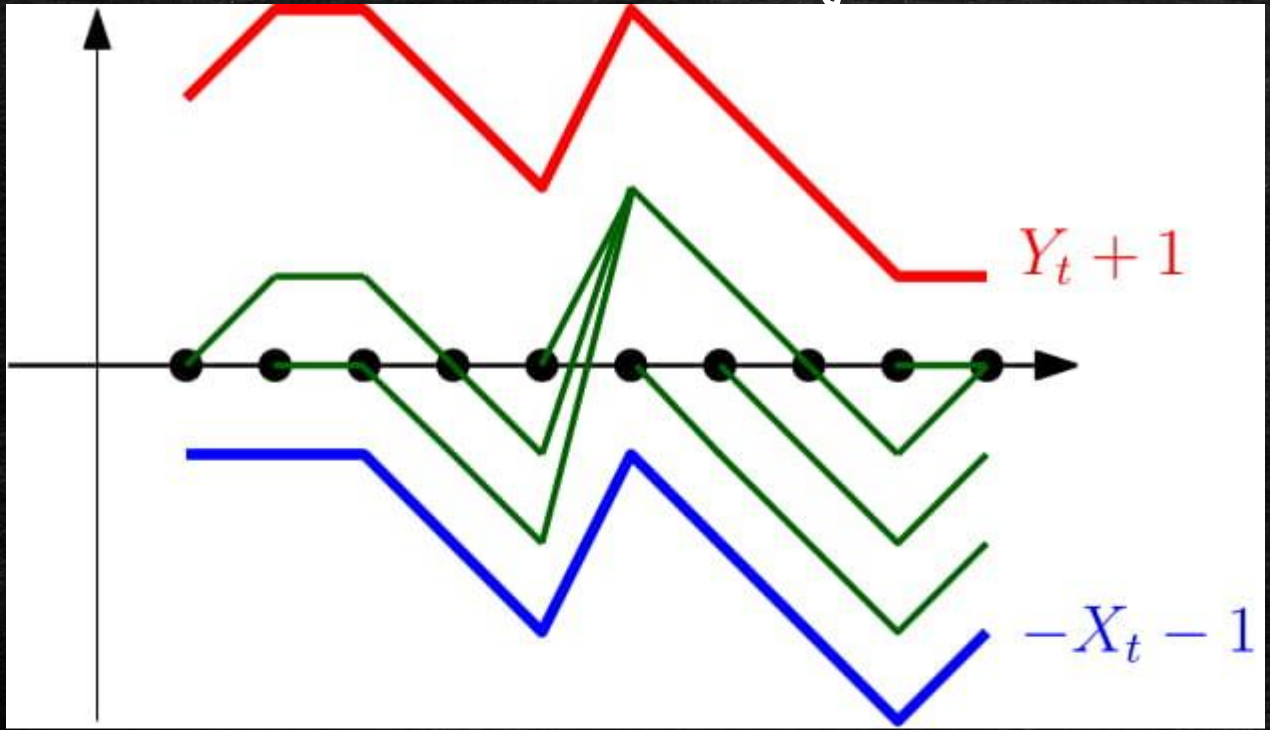
SOLUTION: **COALESCENT - WALK PROCESSES**

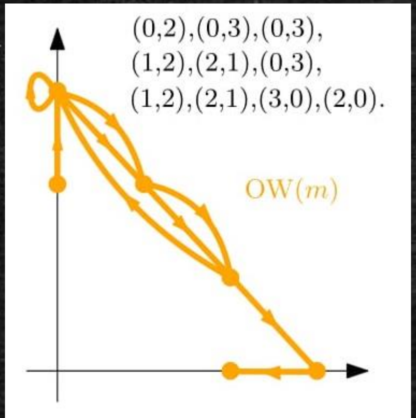
i.e. a collection of walks $\left(Z^{(t)}_{i\geq t}\right)_t$ that "follow" Y_t when they are positive and $-X_t$ when they are negative.

Def: Let $(W_t)_{t\in[n]} = (X_t, Y_t)_{t\in[n]}$ be a tandem walk of length $n\in\mathbb{N}$. The <u>COALESCENT-WALK PROCESS</u> associated to $(W_t)_{t\in[n]}$ is a collection of n one-dimensional walks $(Z^{(t)})_{t\in[n]} =: WC(W)$ defined for every $t\in[n]$ by:

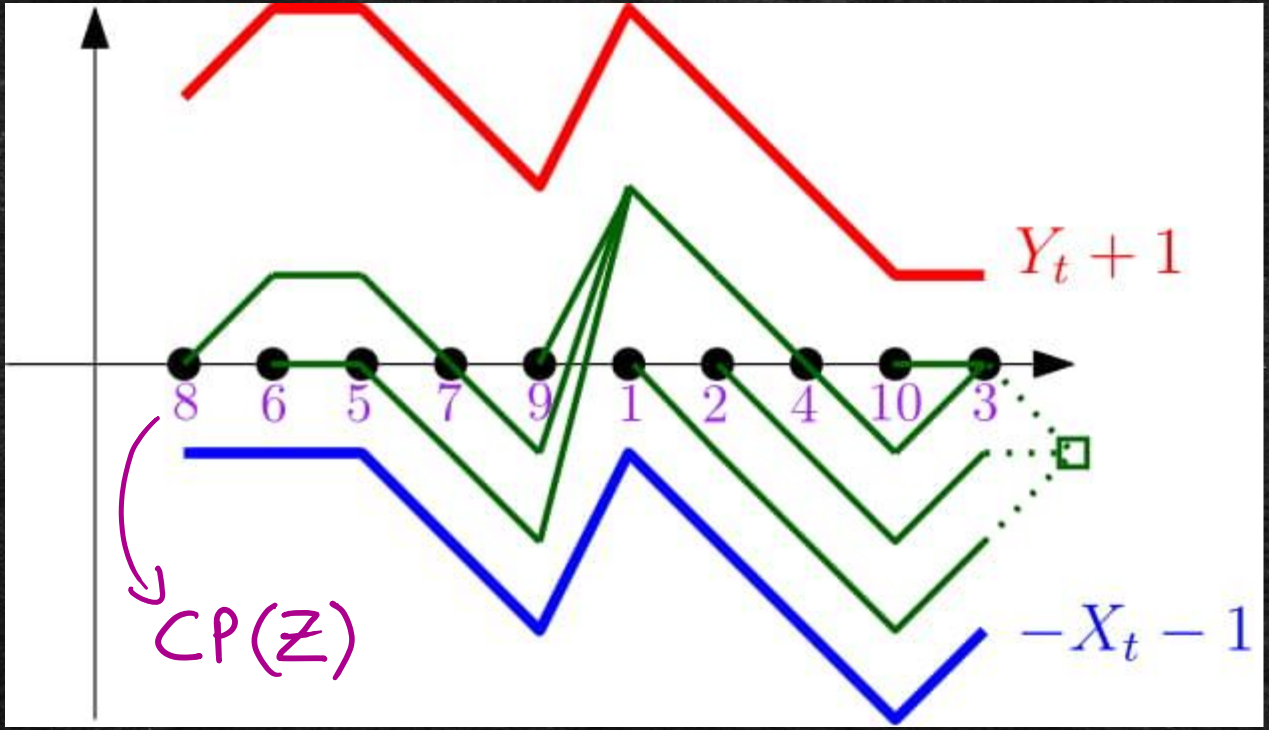
- $Z^{(t)}_t = 0$
- $Z^{(t)}_k = \begin{cases} Z^{(t)}_{k-1} + (Y_k - Y_{k-1}) & \text{if } Z^{(t)}_{k-1} \geq 0 \\ Z^{(t)}_{k-1} - (X_k - X_{k-1}) & \text{if } Z^{(t)}_{k-1} < 0 \;\&\; Z^{(t)}_{k-1} - (X_k - X_{k-1}) < 0 \\ Y_k - Y_{k-1} & \text{if } Z^{(t)}_{k-1} < 0 \;\&\; Z^{(t)}_{k-1} - (X_k - X_{k-1}) \geq 0 \end{cases}$

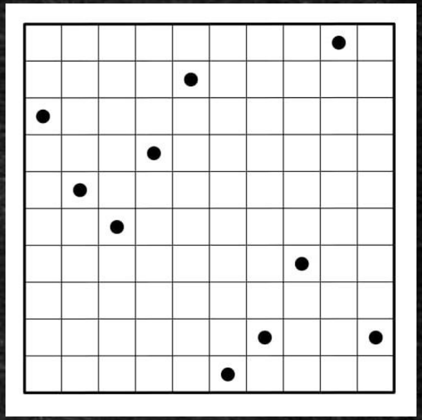






$W = (W_t)_t$

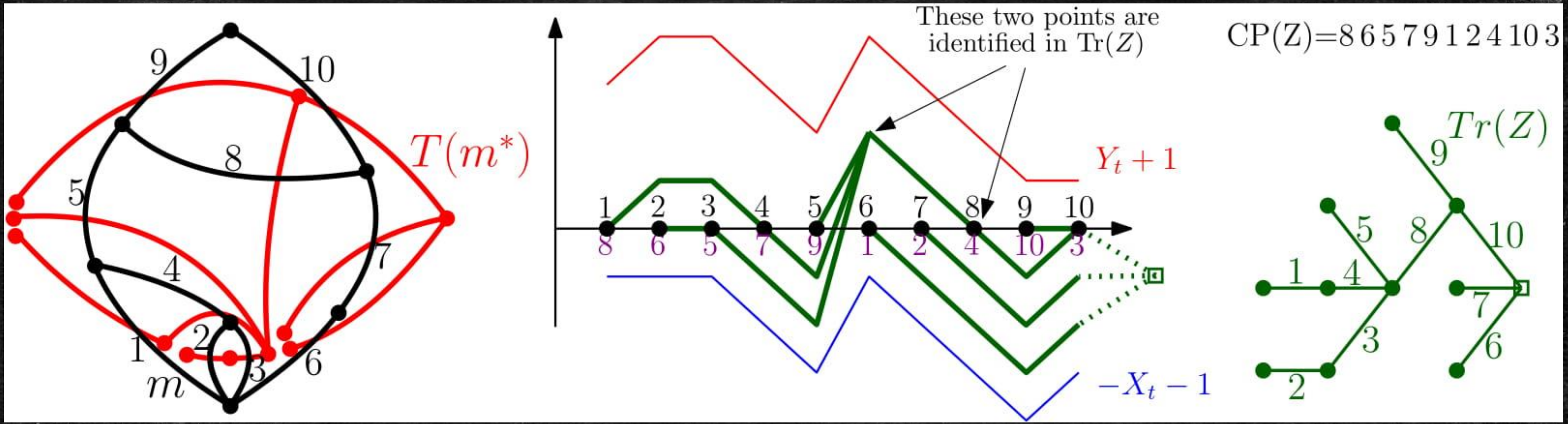




$\sigma = OP \circ OW^{-1}(W)$

THEOREM: Let $W = (W_t)_{t \in [n]}$ be a tandem walk and $\sigma = OP \circ OW^{-1}(W)$ the corresponding Baxter permutation. Then

$$CP \circ \underbrace{WC(W)}_{\text{coalescent-walk process } Z} = \sigma .$$



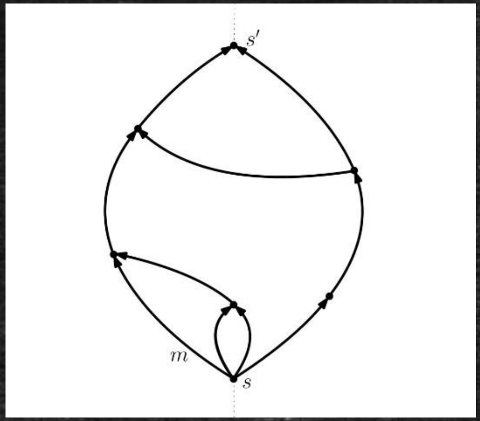
PROPOSITION: Let σ be a Baxter permutation of size $n \in \mathbb{N}$ corresponding to a coalescent-walk process $(Z^{(t)})_{t\in[n]}$. Then for $i<j$

$$\sigma(i) < \sigma(j) \iff Z^{(i)}_j < 0$$

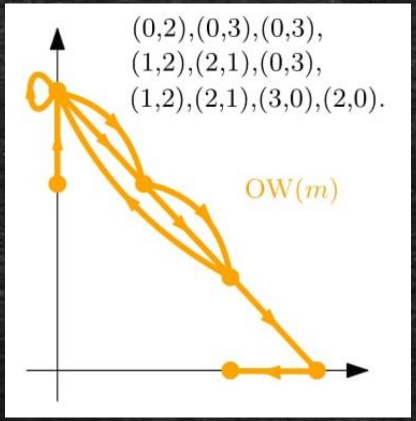
THEOREM: Let m be a bipolar orientation with n edges and $Z = (Z^{(t)})_{t\in[n]}$ be the corresponding coalescent-walk process. Then

$$Tr(Z) = T(m^*) \text{ as labeled trees.}$$

THEOREM: (B.-Maazoun 2020)

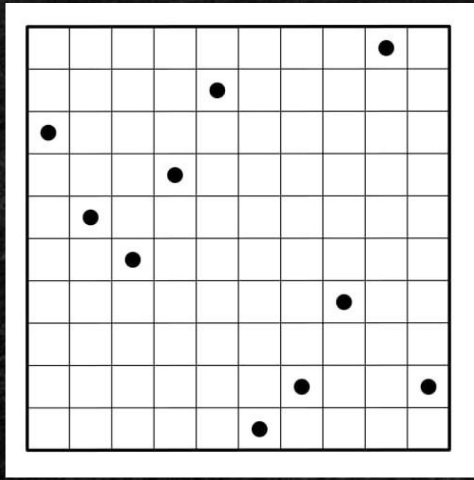


OW

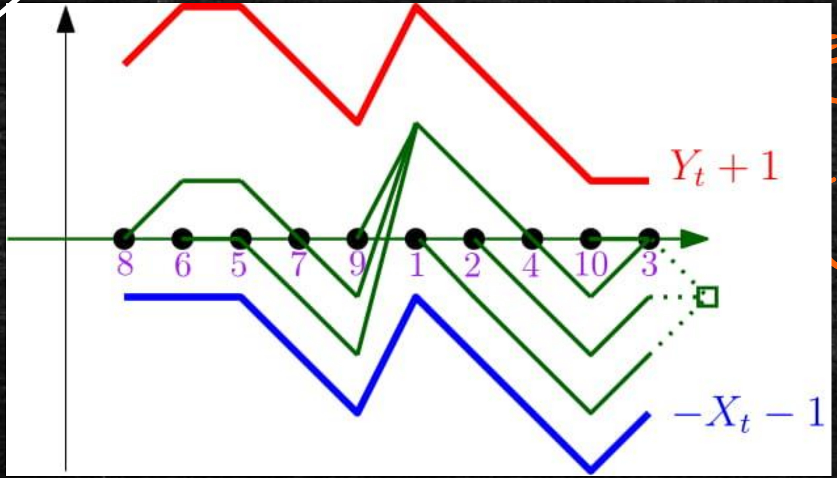


OP

WC



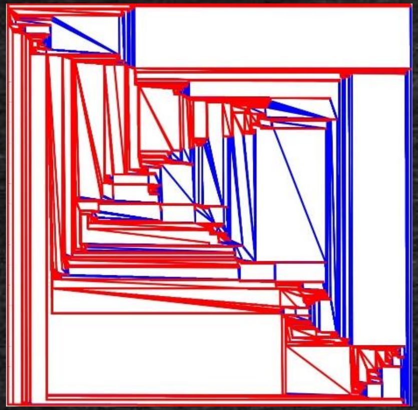
CP



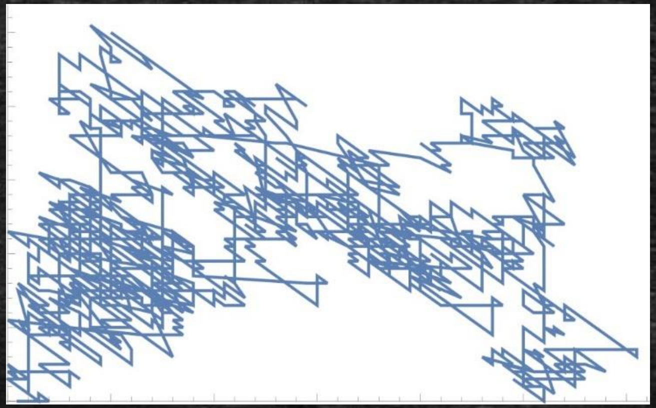
WE WANT "TO READ"
...ERNS IN A
...MUTATION
... CORRESPONDING
...WALK

This diagram commutes.

THEOREM : (B.-Maazoun 2020)

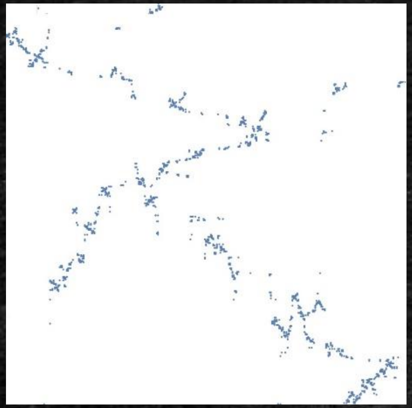


OW →

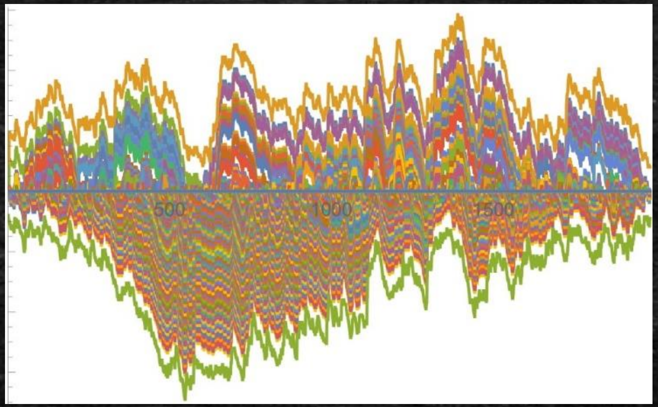


↓ OP

↓ WC



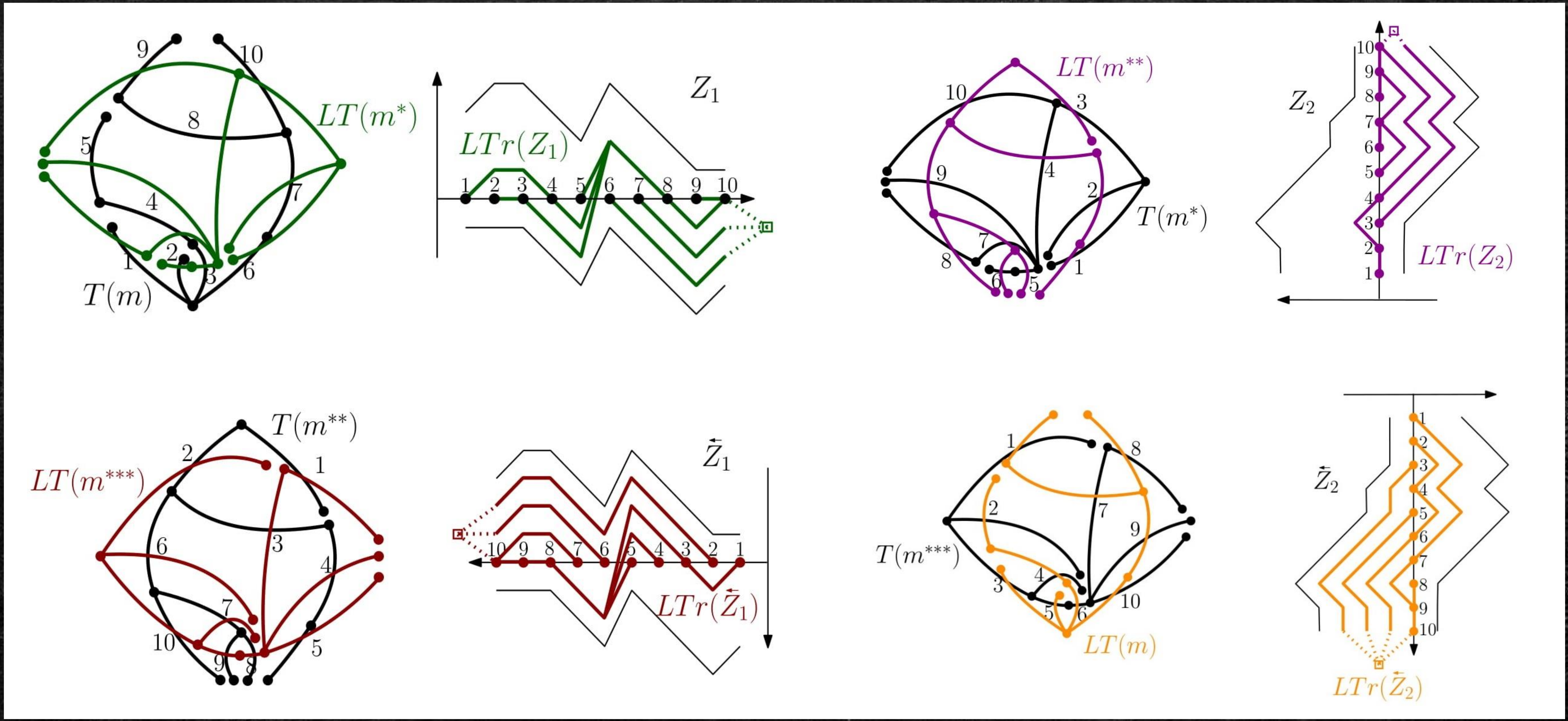
← CP



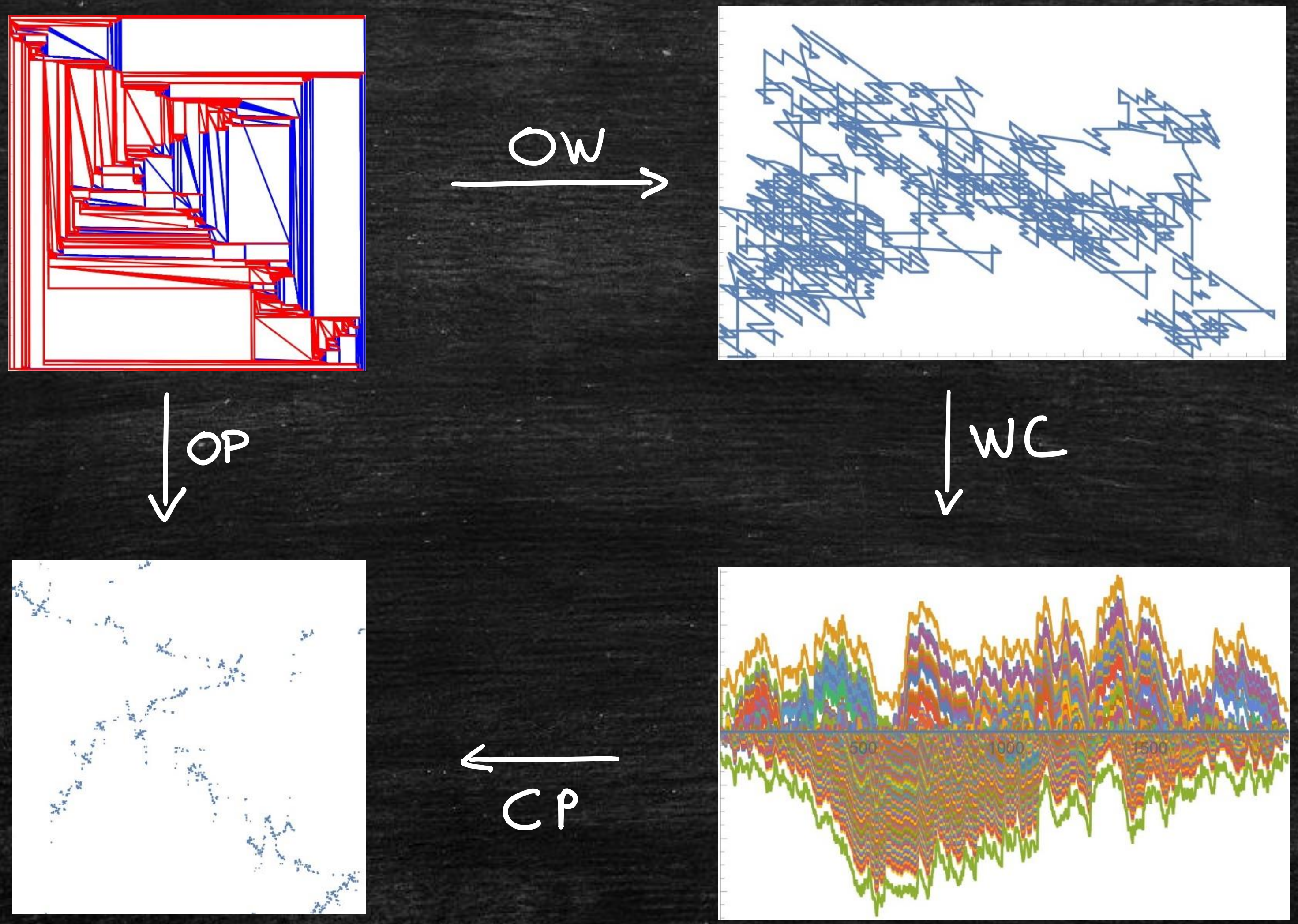
This diagram commutes.

THEOREM : (B.-Maazoun 2020)

$$X_i^* = L_{Z_1}^{(\sigma^{-1}(i))}(n) - 1 \qquad Y_i^* = L_{\overleftarrow{Z}_1}^{(\sigma^{*}(i))}(n) - 1$$



$$L_Z^{(i)}(j) := \#\left\{ k \in [i,j] \mid Z_k^{(i)} = 0 \right\}$$


OW
OP
WC
CP

Scaling limits of coalescent-walk processes

The continuous coalescent-walk process

Consider a two dimensional process $W(t)=(X(t),Y(t))_{t\in I}$ and the following family of stochastic differential equations (SDEs) indexed by $u\in I$

$$(*)\quad \begin{cases} dZ^{(u)}(t) = \mathbb{1}_{\{Z^{(u)}(t)>0\}}\,dY(t) - \mathbb{1}_{\{Z^{(u)}(t)\leq 0\}}\,dX(t), & t\in(u,\infty)\cap I, \\ Z^{(u)}(t)=0, & t\in(-\infty,u]\cap I. \end{cases}$$

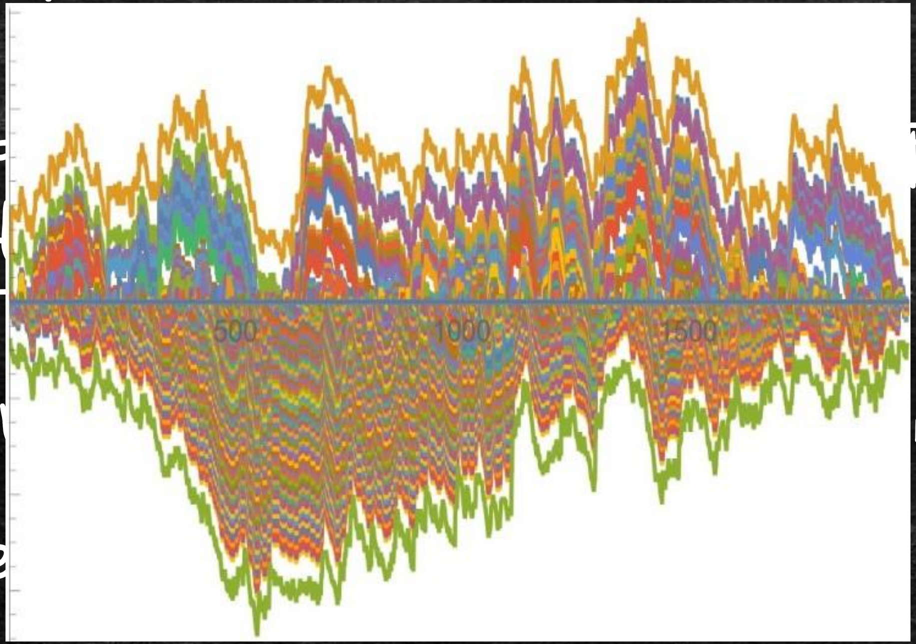
THEOREM (Prokaj 2013, Çağlar-Hajri-Karakus 2018)

Let $(W(t))_{t\in I}$ be a two-dimensional Brownian motion with covariance matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ for some $\rho\in(-1,1)$. Fix $u\in I$. We have path-wise uniqueness and existence of a strong solution for the SDE $(*)$ driven by $W(t)$.

We now consider the SDEs (*) driven by a two-dimensional <u>Brownian excursion</u> $W_e=(X_e,Y_e)$ with cov. matrix $\begin{pmatrix}1 & -\frac{1}{2}\\ -\frac{1}{2} & 1\end{pmatrix}$:

$$\begin{cases} dZ_e^{(u)}(t)=\mathbb{1}_{\{Z_e^{(u)}(t)>0\}}\,dY_e(t)-\mathbb{1}_{\{Z_e^{(u)}(t)\le 0\}}\,dX_e(t), & t>u,\\ Z_e^{(u)}(t)=0, & t\le u,\end{cases}\quad \forall u\in[0,1]$$

Using absolutely co ... proved <u>existence</u> &
<u>uniqueness</u> of s ...



The above ... mply that:

For almost e ... for almost every $u\in[0,1]$.

Def: We call <u>CONTINUOUS COALESCENT-WALK PROCESS</u> (driven by W_e) the collection of solutions $\{Z_e^{(u)}\}_{u\in[0,1]}$ where properly defined.

Let $\overline{W} = (\overline{X}, \overline{Y}) = (\overline{X}_k, \overline{Y}_k)_{k\geq 0}$ be a two dimensional random walk having value $(0,0)$ at time 0 and step distribution

$$\nu = \frac{1}{2}\delta_{(+1,-1)} + \sum_{i,j\geq 0} 2^{-i-j-3}\delta_{(-i,j)}$$

Proposition: The following is a uniform tandem walk of lenght n:

$$(W_t)_{1\leq t\leq n} := \left((\overline{W}_t)_{1\leq t\leq n} \,\middle|\, \overline{W}_0 = (0,0),\ \overline{W}_{n+1} = (0,0),\ (\overline{W}_t)_{0\leq t\leq n+1} \in (\mathbb{Z}^2_{\geq 0})^{n+1} \right)$$

Let $Z = WC(W)$ be the corresponding coalescent-walk process and $(L^{(i)}(j))_{-\infty\leq i\leq j\leq\infty}$ be the corresponding local time process. We define the rescaled continuous versions: for all $n\geq 1$, $u\in\mathbb{R}$, let

$$\mathcal{W}_n : \mathbb{R} \to \mathbb{R}^2 \qquad \mathcal{Z}_n^{(u)} : \mathbb{R} \to \mathbb{R} \qquad \mathcal{L}_n^{(u)} : \mathbb{R} \to \mathbb{R}$$

be the continuous functions defined by interpolating the following points:

$$\mathcal{W}_n\left(\frac{k}{n}\right) = \frac{1}{\sqrt{2n}} W_k \qquad \mathcal{Z}_n^{(u)}\left(\frac{k}{n}\right) = \frac{1}{\sqrt{2n}} Z_k^{(\lceil nu\rceil)} \qquad \mathcal{L}_n^{(u)}\left(\frac{k}{n}\right) = \frac{1}{\sqrt{2n}} L^{(\lceil nu\rceil)}(k), \quad k\in\mathbb{R}$$

THEOREM (B.–Mazzoun 2020): Let $\mu \in (0,1)$. We have the following joint convergence in $\mathcal{C}([0,1],\mathbb{R})^3 \times \mathcal{C}([0,1),\mathbb{R})$

$$\left(\mathcal{W}_n, \mathcal{Z}_n^{(\mu)}, \mathcal{L}_n^{(\mu)}\right) \xrightarrow[n\to\infty]{d} \left(\mathcal{W}_e, \mathcal{Z}_e^{(\mu)}, \mathcal{L}_e^{(\mu)}\right)$$

- $\mathcal{W}_e$: 2-dim. Brownian excursion in the quadrant with cov $\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$
- $\mathcal{Z}_e^{(\mu)}$: associated continuous coal-walk process
- $\mathcal{L}_e^{(\mu)}$: associated local time

Remarks:

- The convergence to the process $\mathcal{W}_e$ is due to Denisov & Wachtel
- The convergence of local times is up to time 1 <u>excluded</u>!

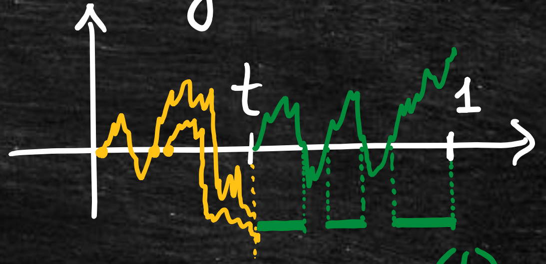
THEOREM (B.–Mazzoun 2020): Let $(\mu_i)_{i\geq 1}$ be a sequence of iid uniform random variables on $[0,1]$ independent of all other variables. Then

$$\left(\mathcal{W}_n, \left(\mathcal{Z}_n^{(\mu_i)}, \mathcal{L}_n^{(\mu_i)}\right)_{i\geq 1}\right) \xrightarrow{d} \left(\mathcal{W}_e, \left(\mathcal{Z}_e^{(\mu_i)}, \mathcal{L}_e^{(\mu_i)}\right)_{i\geq 1}\right)$$

Scaling limits of Baxter permutations

Let $\mathcal{Z}_e = \{\mathcal{Z}_e^{(u)}\}_{u\in(0,1)}$ be the family of solutions of the SDEs $(*)$ driven by the Brownian excursion $\mathcal{W}_e$ in the quadrant of cov. matrix $\begin{pmatrix} -1/2 & 1 \\ 1 & -1/2 \end{pmatrix}$.

Define the following random function for $t\in[0,1]$:



$$\varphi_{\mathcal{Z}_e}(t) := \mathrm{Leb}\left(\{x\in[0,t) \mid \mathcal{Z}_e^{(x)}(t)<0\}\right) + \mathrm{Leb}\left(\{x\in[t,1] \mid \mathcal{Z}_e^{(t)}(x)\geq 0\}\right)$$

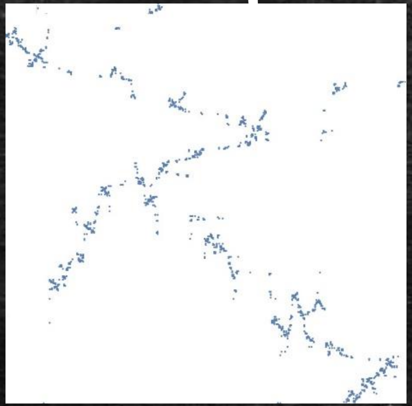
The **BAXTER PERMUTON** is the following random probability measure on the square $[0,1]^2$:

$$\mu_{\mathcal{Z}_e}(\cdot) := (\mathrm{Id},\varphi_{\mathcal{Z}_e})_* \mathrm{Leb}(\cdot) = \mathrm{Leb}\left(\{t\in[0,1] \mid (t,\varphi(t))\in \cdot\}\right).$$

PROPOSITION: $\mu_{\mathcal{Z}_e}(\cdot)$ is almost surely a permuton.

THEOREM (B.-Maazoun 2020)**:** Let σ_n be a uniform Baxter permutation of size n. We have the following convergence in the space of permutons

$$\mu_{\sigma_n} \xrightarrow{d} \mu_{\mathcal{Z}_e}.$$



Proof based on:

- **PROPOSITION:** $\sigma(i) < \sigma(j) \iff Z_j^{(i)} < 0$
- **THEOREM:** $\left(W_n, \left(Z_n^{(u_i)}, L_n^{(u_i)} \right)_{i \geq 1} \right) \xrightarrow{d} \left(\mathcal{W}_e, \left(\mathcal{Z}_e^{(u_i)}, \mathcal{L}_e^{(u_i)} \right)_{i \geq 1} \right)$

Joint scaling limits: the four trees of bipolar orientations

Recall that $m, m^{*}, m^{**}, m^{***}$ are the four bipolar orientations obtained by the duality operation.

Let m be a uniform bipolar orientation with n edges.

From now $\theta \in \{\emptyset, *, **, ***\}$. Given m^{θ} we denote by

W_n^{θ} the corresponding tandem walk, Z_n^{θ} the corresponding coalescent-walk process

L_n^{θ} the corresponding local time process & σ_n^{θ} the corresponding Baxter permutation.

Moreover $\mathcal{W}_n^{\theta}, \mathcal{Z}_n^{\theta}, \mathcal{L}_n^{\theta}$ denote the continuous interpolating versions.

Finally, $(u_{n,i}^{\theta})_{i\geq 1}$ denotes a seq of unif. r.v. on $[0,1]$.

THEOREM (B.-Maazoun 2020): Let u denote a uniform r.v. on $[0,1]$ independent of $\mathcal{W}_e$. Then

1. Almost surely $\mathcal{L}_e^{(u)}$ has a limit at 1 and we still denote its extension by $\mathcal{L}_e^{(u)} \in \mathcal{C}([0,1],\mathbb{R})$.

2. There exists a measurable map $r : \mathcal{C}([0,1],\mathbb{R}^2) \to \mathcal{C}([0,1],\mathbb{R}^2)$ such that almost surely, denoting $(\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}) = r(\mathcal{W}_e)$,

$s(X_t, Y_t) = (Y_{1-t}, X_{1-t})$

$$\tilde{\mathcal{X}}(\varphi_{\mathcal{Z}_e}(u)) = \mathcal{L}_e^{(u)}(1) \quad \text{and} \quad r(s(\mathcal{W}_e)) = s(r(\mathcal{W}_e))$$

These properties uniquely determine the map r $\mathbb{P}_{\mathcal{W}_e}$-a.s. and moreover

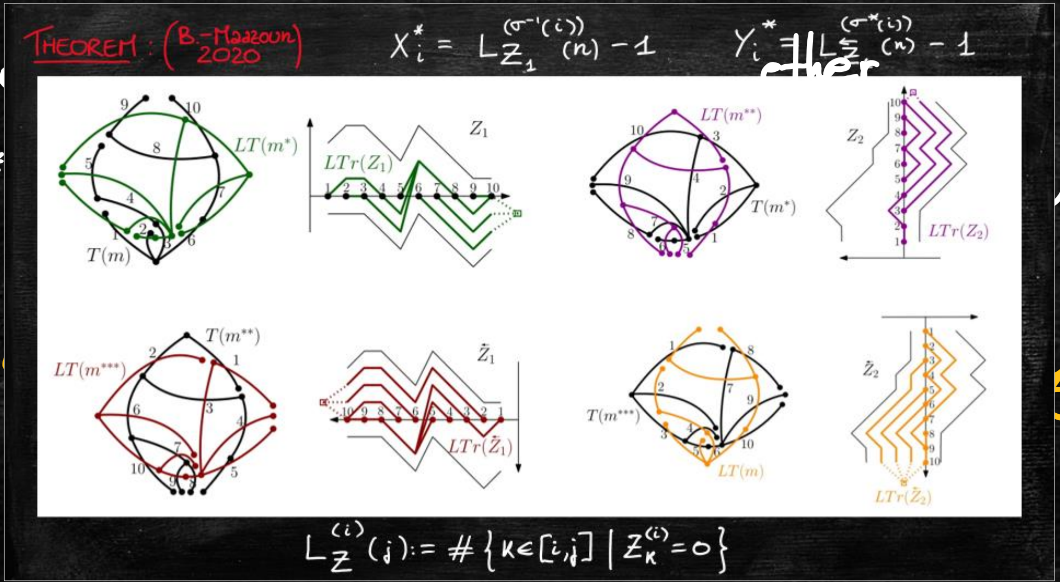
$$r(\mathcal{W}_e) \overset{d}{=} \mathcal{W}_e \;, \quad r^2 = s \;, \quad r^4 = \mathrm{Id} \quad \text{a.s.}$$

3. If we coupl… …verging θ by tacking … other

$\mathcal{W}_e^{\theta *}$ … $u_i^\theta)$ for all $i \geq 0$

then

$\Big(\mathcal{W}_n^\theta, \big(u_{n,i}^\theta, \mathcal{Z}_n^{\theta(u_{n,}}$ … $\mathcal{L}_e^{\theta(u_i^\theta)}, \mathcal{L}_e^{\theta(u_i^\theta)}\big)_{i\geq 1}, u_{\mathcal{Z}_e^\theta}\Big)_\theta$



Future projects

Fix $\rho \in [-1,1]$ and $q \in [0,1]$. Consider a two-dim. Brow. excursion $\mathcal{E}_\rho = (X_\rho, Y_\rho)$ with cov. matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ & the SDEs

$$(*) \begin{cases} dZ^{(u)}_{\rho,q}(t) = \mathbb{1}_{\{Z^{(u)}_{\rho,q}(t)>0\}} dY_\rho(t) - \mathbb{1}_{\{Z^{(u)}_{\rho,q}(t)\leq 0\}} dX_\rho(t) + (2q-1)\, d\mathcal{L}^Z(t), & t > u \\ Z_{\rho,q}(t) = 0, & t \leq u \end{cases}$$

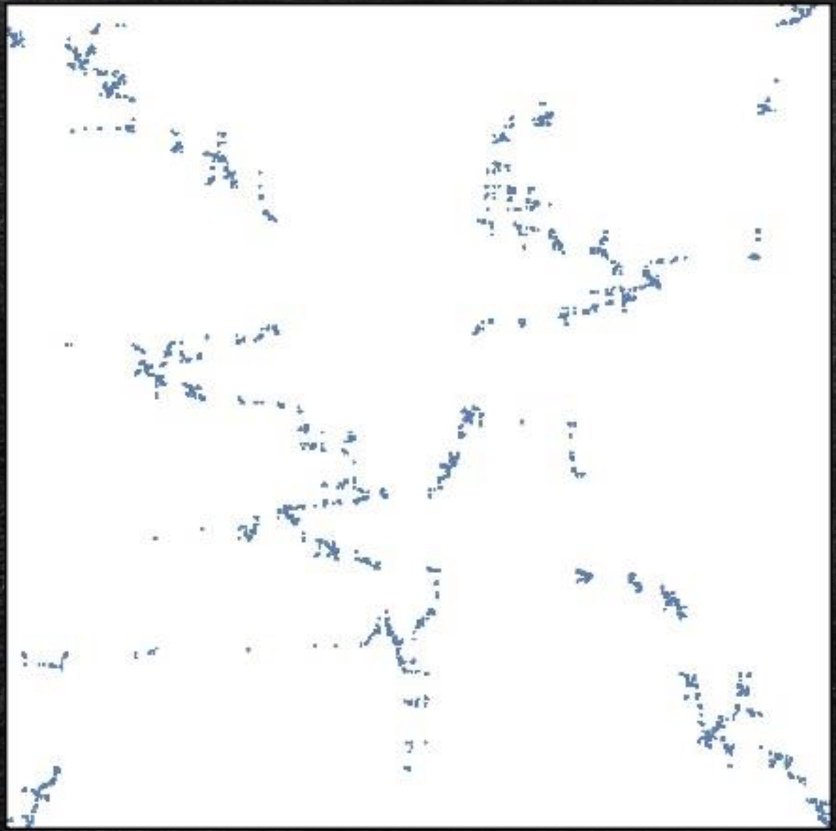
where $\mathcal{L}^Z(t)$ is the local time process of $Z^{(u)}_{\rho,q}(t)$ at zero.

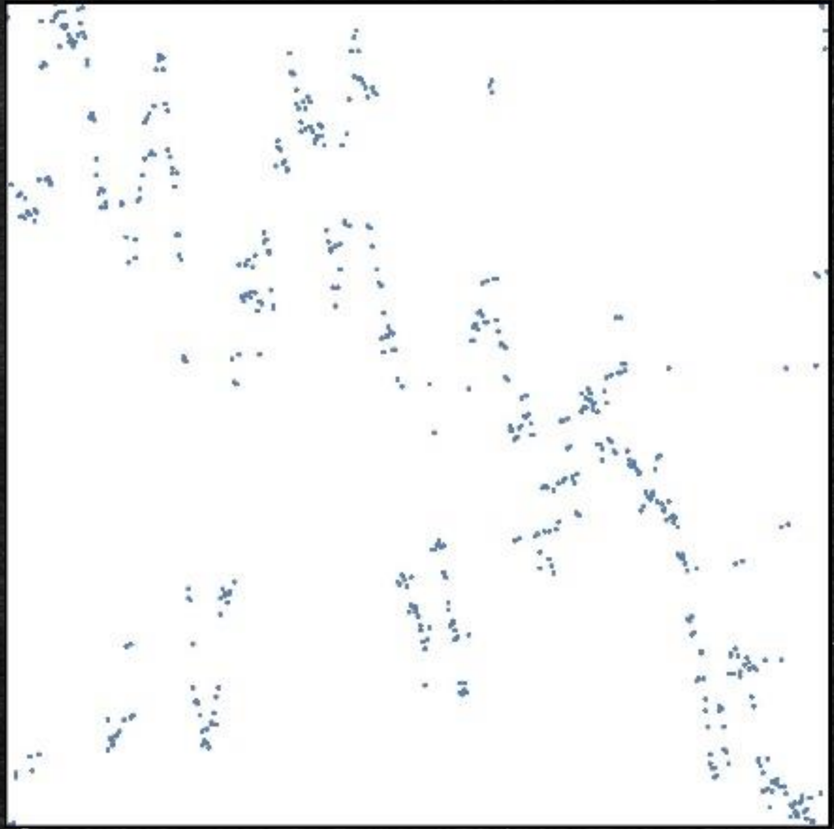
From $(*)$ we can define $\mu_{\rho,q} := \mu_{Z_{\rho,q}}$. $\left[\text{BAXTER PERMUTON} = \mu_{-\frac{1}{2},\frac{1}{2}}\right]$

CONJECTURE: The Brownian sep. permuton μ_p satisfies

$$\mu_p \overset{d}{=} \mu_{-1,1-p}.$$

The permuton $\mu_{\rho,q}$ is a NEW UNIVERSAL LIMITING OBJECT.



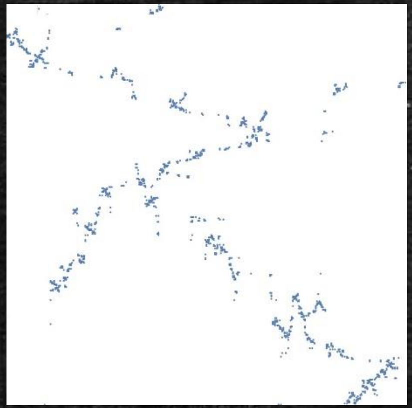
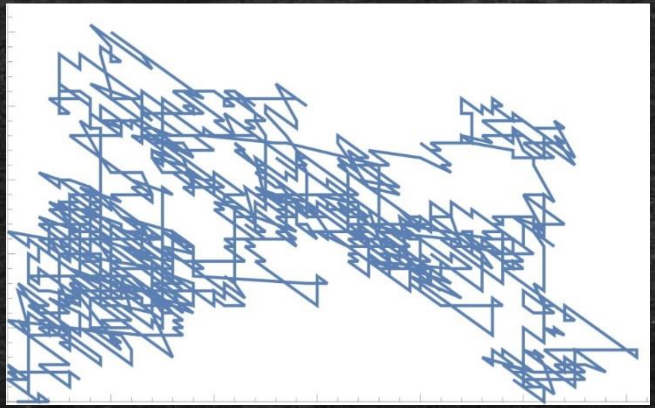
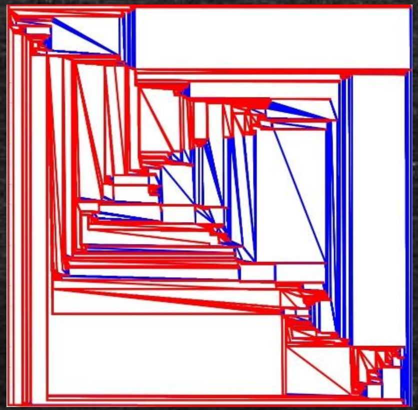
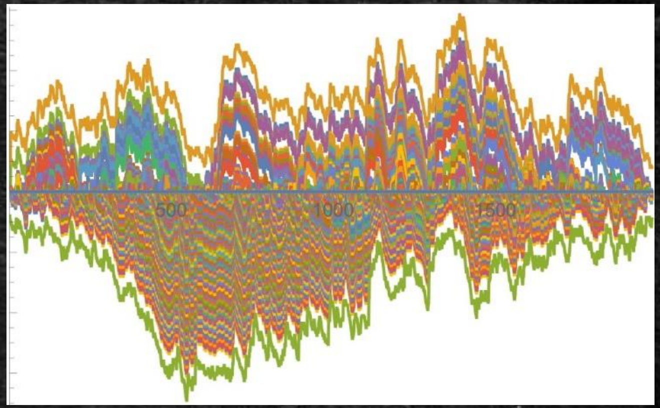


Baxter $\downarrow$ $\mathcal{M}_{-\frac{1}{2},\frac{1}{2}}$

CONJ: Semi-Baxter $\downarrow$ $\mathcal{M}_{\rho,q}$ with $|\rho|<1$, $q\neq\frac{1}{2}$

Final comments

- Our result implies convergence of finite-volume bip-orientations to a $\sqrt{4/3}$-LQG. What is the connection between our approach and the LQG approach?
- We also proved joint Benjamini-Schramm local limits (both in the ANNEALED & QUENCHED sense) for all the objects involved in the commutative diagram.
- We believe that our techniques are rather general: we would like to consider other families of permutations (and maps?) encoded by two-dimensional walks.
- We would also like to investigate better the generalized Baxter permuton. For instance, what is $\mathbb{E}[\mu_{\rho,q}] = ?$.
- Relations between the parameters q and θ (of LQG)?

THANK YOU!

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