Solutions 9

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Exercise 1

1. Consider $A = \{2^0, 2^1, 2^2, \dots, 2^{k-1}\}$. Then $|A| = k$ and if $k = \lfloor \log_2(n) + 1 \rfloor$ then $A \subseteq [n]$. Note also that A is sum-free since the elements of A can be thought as the representations in base 2 of numbers. We conclude that

$$K(n) \geq \log_2(n)$$

2. Fix a set $A = \{a_1, \dots, a_k\} \subseteq [n]$. Note that the number of possible sums of the form $\sum_{i \in S} a_i$, with $S \subseteq [k]$, is 2^k. Note also that $\max_{S \subseteq [k]} \{\sum_{i \in S} a_i\} \leq k \cdot n$.
Therefore if we want that the sums are distinct we need that

$$2^k \leq k \cdot n$$

Since this bound is true $\forall$ A of cardinality k, we conclude that

$$2^{K(n)} \leq K(n) \cdot n$$

3. Note that if $k > \log_2(n) + \log_2(\log_2(n)) + cost$, i.e.

$$k = \log_2(n) + \log_2(\log_2(n)) + cost + h, \text{ for some } h \in \mathbb{N},$$

then

$$2^k = 2^{\log_2(n) + \log_2(\log_2(n)) + cost + h} = n \log_2(n)\, 2^{cost + h}$$

and

$$k \cdot n = n\left(\log_2(n) + \log_2(\log_2(n)) + cost + h\right)$$

Therefore $2^k > k \cdot n \iff \log_2(n)\, 2^{cost+h} > \log_2(n) + \log_2(\log_2(n)) + cost + h$

and the last inequality is true for all $h \in \mathbb{N}$ setting (for example) $cost = 2$.

Therefore it has to hold that $k(n) \leq \log_2(n) + \log_2(\log_2(n)) + cost.$

4. We fix $A = \{a_1, \dots, a_k\} \subseteq [n]$ and we consider

$$X = \sum_{i=1}^{k} \varepsilon_i a_i .$$

Note that

$$\mathbb{E}[X] = \sum_{i=1}^{k} a_i/2$$

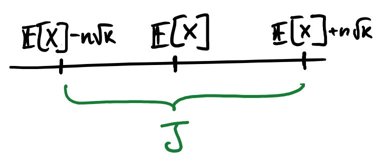
$$Var[X] = \sum_{i=1}^{k} a_i^2 \, Var(\varepsilon_i) = \sum_{i=1}^{k} a_i^2/4$$

By Chebishev's inequality $\left(\mathbb{P}\left(|X - \mathbb{E}[X]| \geq \lambda \sqrt{Var(X)}\right) \leq \frac{1}{\lambda^2}\right)$ we have that

$$\mathbb{P}\left(|X - \mathbb{E}[X]| \geq n\sqrt{k}\right) \underset{\lambda = \frac{n\sqrt{k}}{\sigma}}{\leq} \frac{Var(X)}{n^2 k} = \frac{\sum_{i=1}^{k} a_i^2/4}{n^2 k} \underset{a_i \leq n \;\; \forall i \leq k}{\leq} \frac{\frac{1}{4} \cdot n^2 k}{n k} = \frac{1}{4}$$

5. Set $J = \left[\mathbb{E}[X] - n\sqrt{k}, \mathbb{E}[X] + n\sqrt{k}\right]$

INTUITION:



Using point 4, we have that

" " "

$\left(\frac{3}{4}\right)^{th}$ of the sums in A are in J

then, since the sums have to be distinct,

$$\frac{3}{4}\underbrace{2^k}_{\text{possible sums}} \leq 2n\sqrt{k}$$

5. Similar as 3.

Exercise 3

$Y := X - m \Rightarrow \mathbb{E}[Y] = 0 \quad \mathbb{E}[Y^2] = Var(X)$

1. $$\mathbb{P}(X \geq \lambda) = \mathbb{P}(X - m \geq \lambda - m) \overset{?}{=} \mathbb{P}(Y + b \geq \lambda - m + b) \quad \forall b \in \mathbb{R}$$

$$\leq \frac{\mathbb{E}[(Y+b)^2]}{(\lambda - m + b)^2} = \frac{\sigma^2 + b^2}{(\lambda - m + b)^2} \leq \frac{\sigma^2}{\sigma^2 + (\lambda - m)^2}$$

Markov
(we need positive variables)

Note that $\min_b \left\{ \frac{\sigma^2 + b^2}{(\lambda - m + b)^2} \right\}$
occours at $b = \frac{\sigma^2}{\lambda - m}$

Exercise 2

We use Borel-Cantelli lemma. Set $Y_n = \frac{T_n}{\mathbb{E}[T_n]} - 1$. We have that:

if $\sum_{n=1}^{\infty} \mathbb{P}(|Y_n| > \varepsilon) < \infty$ then $Y_n \xrightarrow{a.s.} 0$

By Chebischev's inequality $\left[\mathbb{P}\left(|X - \mathbb{E}[X]| \geq \lambda\sqrt{Var(X)}\right) \leq \frac{1}{\lambda^2}\right]$ we have that

$$\mathbb{P}(|Y_n| > \varepsilon) \overset{\lambda = \varepsilon/\sqrt{Var(Y_n)}}{\leq} \frac{Var(Y_n)}{\varepsilon^2}$$

We now note that

$$Var(Y_n) = \frac{1}{\mathbb{E}[T_n]^2} Var(T_n) \overset{\text{lecture}}{=} O\left(\underbrace{\frac{n^4 p^5}{n^6 p^6}}_{1/n^2} + \underbrace{\frac{n^3 p^3}{n^6 p^6}}_{\frac{1}{n^3}} \right)$$

Since both terms are summable we can easly conclude.