Random combinatorial structures
Exercise sheet nb. 9

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Exercise 1. We say that a set $A = \{a_1, \dots, a_k\} \subset [n]$ is *sum-free* if for all $S \subset [k]$ the sums $\sum_{i \in S} a_i$ are distinct. Let $K(n)$ denotes the maximal cardinality of a sum-free set contained in $[n]$.

1. Show that for every $n \geq 1$,
$$K(n) \geq \log_2(n).$$
2. Show that for every $n \geq 1$, $K(n)$ satisfies the inequality
$$K(n) \cdot n \geq 2^{K(n)}.$$
3. Deduce that
$$K(n) \leq \log_2(n) + \log_2(\log_2(n)) + cost.$$
4. Fix a sum-free set $A = \{a_1, \dots, a_k\} \subset [n]$ of cardinality k. Consider now the following random variable
$$X = \sum_{i=1}^{k} \varepsilon_i a_i,$$
where ε_i are i.i.d. Bernoulli random variables of parameter $1/2$. Show that
$$\mathbb{P}(|X - \mathbb{E}[X]| \geq n\sqrt{k}) \leq \frac{1}{4}.$$
5. Deduce that
$$\frac{3}{4} 2^k \leq 2n\sqrt{k}.$$
6. Conclude that
$$K(n) \leq \log_2(n) + \frac{1}{2}\log_2(\log_2(n)) + cost.$$

Exercise 2. Let $G(n, p_n)$ the Erdos–Rényi graph. In class we saw that as $np_n \to \infty$ then $G(n, p_n)$ contains a triangle with probability tending to one.

Show that when $p \in (0, 1)$ is fixed and T_n denotes the number of triangles in $G(n, p)$, then

$$\frac{T_n}{\mathbb{E}[T_n]} \xrightarrow{a.s.} 1.$$

Exercise 3 (One-side bound). Let X be a random variable with expectation m and variance σ^2.

1. Show that

$$P(X \geq \lambda) \leq \frac{\sigma^2}{\sigma^2 + (\lambda - m)^2}, \quad \text{for all} \quad \lambda \geq m.$$

   Hint: Apply Markov to $P(f(X) \geq f(\lambda))$ for a well-chosen linear function f.

2. Compare with Chebishev's inequality for $P(|X - m| \geq \lambda - m)$.