Random combinatorial structures
Exercise sheet nb. 10

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Exercise 1. We consider the following situation: we have two uniformly shuffled decks with $4n$ cards (n for each of the 4 possible suits). We turn the card of the two decks one by one (one from each deck at each time) and we count the number of times you get the same value, but possibly with different suits. We denote this number as X_n.

1. Note that X_n is distributed as the number of i such that $\sigma(i) \equiv i \mod n$ in a uniform permutation σ of size $4n$.

2. Write X_n as a sum of indicator functions.

3. We recall that factorial moments of sums of indicator variables write nicely:
$$\mathbb{E}\left[\left(\sum_{\alpha \in A} I_\alpha\right)_r\right] = \sum_{\substack{\alpha_1,\dots,\alpha_r \in A \\ \textbf{distinct}}} \mathbb{E}\big(I_{\alpha_1}\dots I_{\alpha_r}\big).$$
Using this result, write the r-th the factorial moment of X_n as:
$$\sum_{\substack{i_1,\dots,i_r \in [4n] \\ \textbf{distinct}}} \mathbb{P}(\sigma(i_1) \equiv i_1 \mod n, ..., \sigma(i_r) \equiv i_r \mod n). \tag{1}$$

4. Show that
$$\mathbb{P}(\sigma(i_1) \equiv i_1 \mod n, ..., \sigma(i_r) \equiv i_r \mod n) \leq \frac{4^r(4n-r)!}{(4n)!}.$$

5. Show that the number of terms in Equation (1) where some i_j's are congruent to each other is $O(n^{r-1})$.

6. Conclude that X_n converges to a Poisson random variable of parameter 4.

Exercise 2. Let X and Y be two real-valued random variables with densities:

$$f_X(x) = \frac{e^{-\log(x)^2/2}}{x\sqrt{2\pi}} \mathbf{1}_{x>0},$$

$$f_Y(x) = \frac{e^{-\log(x)^2/2}}{x\sqrt{2\pi}} \big(1 + \sin(2\pi \log(x))\big) \mathbf{1}_{x>0}.$$

(Why does this define random variables?)

Show that X and Y have the same moments but distinct distributions.

Hint: compute the r-th moments of X and Y as an integral and do the change of variables $x = \log(t + r)$.