Exercise 1

Lemma: Let $\mathcal{C}$ be a combinatorial class without elements of size 0 and set $\mathcal{A} = \mathrm{Seq}(\mathcal{C})$. Then

$$A(z,u) = \frac{1}{1-C(z,u)}.$$

Proof: $A(z,u) = 1 + C(z,u) + C^2(z,u) + \dots = \dfrac{1}{1-C(z,u)}$ $\square$

lemma for the product of classes

Remark: $(1-C(z,u))$ is invertible as formal power series since $C(z,u)$ has no constant term.

Indeed if f is a FPS with non-zero constant term, then then f has an inverse for the multiplication:

Write $f = a + zg$ with $a \neq 0$, g a FPS.

$\dfrac{1}{1+z} = \sum_{n\geq 0} (-1)^n z^n$ is a FPS

Consider the FPS $h = a^{-1} \dfrac{1}{1+(a^{-1}zg)}$

inverse of a

a FPS because it is an admissible composition "$\frac{1}{1+x} \circ (a^{-1}zg)$"

$$hf = \cancel{a^{-1}} \frac{1}{1+(a^{-1}zg)} \cancel{a} (1+a^{-1}zg) = \left(\sum_{n\geq 0} (-1)^n (a^{-1}zg)^n\right)(1+a^{-1}zg)$$

$$= \sum_{n\geq 0} (-1)^n (a^{-1}zg)^n + \sum_{n\geq 0} (-1)^n (a^{-1}zg)^{n+1}$$

$$= 1 + \sum_{n\geq 1} (-1)^n (a^{-1}zg)^n - \sum_{n\geq 0} (-1)^{n+1} (a^{-1}zg)^{n+1} = 1$$ $\square$

Exercise 2

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1) We denote $\mathcal{C}^n := \{\text{Composition of size } n\}$

Any composition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n in k parts can be represented with n balls and $k-1$ bars (where the i-th bar is after the $\left(\sum_{j=1}^{i} \lambda_j\right)^{th}$ balls). We denote this representation with $\varphi(\lambda)$.

Example: $10 = (2, 3, 1, 4)$ is represented by $\bullet\bullet | \bullet\bullet\bullet | \bullet | \bullet\bullet\bullet\bullet$

Claim: φ is a bijection from $\mathcal{C}^n$ to $\mathcal{B}^n$ $\left\{\begin{array}{l}\text{Sequences of } n \text{ balls and} \\ 0 \le k \le n-1 \text{ bars s.t. there are no consecutive} \\ \text{bars and no bars at the beginning and} \\ \text{at the end}\end{array}\right\}$

NOT NEEDED!

Proof: $\varphi(\mathcal{C}^n) \subseteq \mathcal{B}^n$ since $\forall \lambda = (\lambda_1, \dots, \lambda_k) \in \mathcal{C}^n$ $\lambda_i > 0$ by def. and so by construction of φ all the bars are non-consecutive and not at the beginning or at the end.
Moreover given $b \in \mathcal{B}^n$ is easy to reconstruct the unique $\lambda \in \mathcal{C}^n$ s.t. $\varphi(\lambda) = b$. Indeed it is enough to set λ_1 to be the number of balls before the first bar, λ_2 the number of balls between the first and the second bar, and so on. $\square$

2) Noting that every sequence in $\mathcal{B}^n$ can be obtained starting with a ball and then adding either a ball or a bar followed by a ball (since there are no consecutive bars), we have

increase the size by 1

$$\mathcal{C}^n \simeq \mathcal{B}^n \simeq \{\bullet\} \times \mathrm{Seq}(\{\bullet, |\bullet\})$$

increse the number of parts by one

3) The BGF for $\{\bullet, |\bullet\}$ is $z + uz$, therefore the BGF for $\{\bullet\} \times \mathrm{Seq}\{\bullet, |\bullet\}$ is $z \cdot \dfrac{1}{1-(z+uz)}$.

Note that we are undercounting the number of parts by 1 (the first ball "corresponds" to the first part) and therefore

$$C(z,u) = \frac{zu}{1 - z(1+u)} = zu \sum_{n \ge 0} (z(1+u))^n = \sum_{n \ge 0} u(1+u)^n z^{n+1}$$

$$\frac{1}{1-x} = \sum x^n$$

$$1-z(1+u)$$

$\frac{1}{1-x} = \sum_{n\geq 0} x^n$

4) We know that if $X_n = \#$ of parts in a unif perm. of size n, the corresp. PGF is

$$P_n(u) = \frac{[z^n]\, C(z,u)}{[z^n]\, C(z,1)} = \frac{u(1+u)^{n-1}}{2^{n-1}}$$

$$\Longrightarrow \quad P_n(u) = A(u)\, B(u)^{\beta_n}$$

with $A(u) = u$, $B(u) = \frac{1+u}{2}$ and $\beta_n = n-1$.

Noting that $A(u)$ & $B(u)$ are holom., $A(1) = B(1) = 1$ and that $\underbrace{B''(1)}_{=0} + \underbrace{B'(1)}_{=\frac{1}{2}} - \underbrace{B'(1)^2}_{=\frac{1}{4}} \neq 0$ ("variability condition") we can conclude that

$U'(0) = B'(1)$

$$E[X_n] = \beta_n U'(0) + O(1) = \frac{n-1}{2} + O(1)$$

$$Var[X_n] = \beta_n U''(0) + O(1) = \frac{n-1}{4} + O(1)$$

$U''(0) = B''(1) + B'(1) - B'(1)^2$.

and that $\frac{X_n - \frac{n-1}{2}}{\frac{\sqrt{n-1}}{2}} \xrightarrow{d} N(0,1)$. $\square$

Exercise 3

We only show the property that we need for the following exercise:

3) $\Gamma(z+1) = \int_0^\infty x^z e^{-x}\, dx \underset{\text{integration by part}}{=} \Big[x^z \overbrace{(-e^{-x})}^{=\int e^{-x}dx}\Big]_{x=0}^{\infty} + \int z x^{z-1} e^{-x}\, dx$

$= z \int x^{z-1} e^{-x}\, dx = z\, \Gamma(z)$. $\square$

Exercise 4

formula seen during the lecture!

formula seen during the lecture!

1) $$\frac{\Gamma(u+n)}{\Gamma(u)\underbrace{\Gamma(n+1)}_{n!}} = \frac{1}{n!}\prod_{i=0}^{n-1}(u+i) = \prod_{i=0}^{n-1}\frac{u+i}{1+i} = P_n(u).$$

$$\Gamma(u+n) = (u+n-1)\,\Gamma(u+n-1) = \dots = (u+n-1)(u+n-2)\cdots u\,\Gamma(u)$$

2) $$\frac{\Gamma(u+n)}{\Gamma(u)\Gamma(n+1)} \sim \frac{1}{\Gamma(u)}\cdot \underbrace{\frac{\sqrt{u+n-1}}{\sqrt{n}}}_{\sim 1}\;\frac{\left(\frac{u+n-1}{e}\right)^{u+n-1}}{\left(\frac{n}{e}\right)^n}$$

$$\sim \frac{1}{\Gamma(u)}\cdot \underbrace{\frac{(u+n-1)^{u+n-1}}{n^n}}\cdot e^{1-u} \sim \frac{1}{\Gamma(u)}\cdot n^{u-1}$$

$$\underbrace{(u+n-1)^{u-1}}_{\sim n^{u-1}} \qquad \underbrace{\left(\frac{u+n-1}{n}\right)^n}_{\left(1+\frac{(u-1)}{n}\right)^n \sim e^{u-1}}$$

$$\Rightarrow \quad P_n(u) \sim \frac{1}{\Gamma(u)}\cdot n^{u-1} = \frac{1}{\Gamma(u)}\cdot \left(e^{u-1}\right)^{\log(n)}$$

$$\Rightarrow \quad P_n(u) = A(u)\,B(u)^{\beta_n}$$

with $A(u) = \frac{1}{\Gamma(u)}$, $B(u) = e^{u-1}$ and $\beta_n = \log(n)$
$(B'(u) = B''(u) = e^{u-1})$

Noting that $A(u)$ & $B(u)$ are holom., $A(1) = B(1) = 1$ and that $\underset{1}{B''(1)} + \underset{1}{B'(1)} - \underset{1}{B'(1)}^2 \neq 0$ ("variability condition") we can conclude that

$$E[X_n] = \beta_n U'(0) + O(1) \overset{U'(0)=B'(1)}{=} \log(n) + O(1)$$

$$Var[X_n] = \beta_n U''(0) + O(1) = \log(n) + O(1)$$

$$U''(0) = B''(1) + B'(1) - B'(1)^2.$$

and that $\dfrac{X_n - \log(n)}{\sqrt{\log(n)}} \xrightarrow{d} N(0,1)$.

(We recovered the result seen in the second lecture of the course with much less work (because now we have the q-p. thm))