Enumerative combinatorics
Mid-term homework

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November 22, 2018

Timeline and pratical information:

You need to return your solution to this homework to Jacopo Borga at the latest on Monday November 19, at 18:00.

Your solution will be graded, and returned to you on November 22 during the exercise session.

The solution to the homework will be discussed during the exercise session of November 22 (instead of the correction of an exercise sheet).

If H (resp. F) is your grade for this homework (resp. the final exam), your final grade for the course will be $\max(F, \frac{2F+H}{3})$.

You are very welcome to hand in a solution to the homework even if you cannot attend the exercise sessions.

Exercise 1. Prove bijectively that there are 2^{n-1} compositions of any $n > 0$.

Exercise 2. The purpose of this exercise is to convince you that even power series with zero radius of convergence can count something interesting and can be very useful.

A permutation π of $[n] = \{1, 2, \ldots, n\}$ (we say that π is a permutation of size n) is an *indecomposable* permutation if there does not exist a number k with $1 < k < n$ such that π permutes the numbers $1, \ldots, k$ among themselves.

We let $P(z) = \sum_{n\geq 0} p_n z^n$ be the ordinary generating function for the class of permutations, *i.e.* $p_n = n!$ for all $n \geq 0$ (recall that $p_0 = 0! = 1$) and $S(z) = \sum_{n\geq 0} s_n z^n$ be the ordinary generating function for the class of indecomposable permutations (by convention $s_0 = 0$).

1. Noting that any permutation σ of size $n \geq 1$ is obtained by stitching together an indecomposable permutation of size $0 < k \leq n$ and an arbitrary permutation of the set $\{k+1, \ldots, n\}$, write a relation for p_n depending on $(s_k)_{0<k\leq n}$.
2. What is the coefficient of z^n in $(1 - S(z)) \cdot P(z)$?
3. Using the "multiplicative inverse formula" determine $S(z)$.

Exercise 3. In this exercise we explore the combinatorial classes of unary-binary trees and of Motzkin paths.

Unary-binary trees are rooted plane trees where every vertex has arity (*i.e,* number of children) 0, 1 or 2. The size of a unary-binary tree is its number of vertices.

1. Find a recursive equation that describes the class $\mathcal{UB}$ of unary-binary trees.

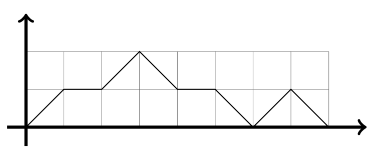
2. Translate it into an equation for the ordinary generating function $\mathrm{UB}(z)$ of unary-binary trees.

3. Solve it to give an explicit expression for $\mathrm{UB}(z)$.

4. Find an expression of the number of unary-binary trees of size n. (Hint: use Lagrange inversion.)

By definition a Motzkin path of size n is a path from $(0,0)$ to $(n,0)$, staying weakly-above the horizontal axis and using the following set of steps:

- up-step (1,1) ;
- down-step (1,-1);
- horizontal step (1,0).

Example:  .

Let $\mathcal{M}$ be the combinatorial class of Motzkin paths.

5. By considering the first return (*i.e.* the smallest number $j > 0$ such that $(j,0)$ is on the path), show that :

$$\mathcal{M} = \mathcal{E} + \mathcal{Z} \times \mathcal{M} + \mathcal{Z}^2 \times \mathcal{M} \times \mathcal{M}, \qquad (\star)$$

where $\mathcal{E}$ is the class with one element of size 0 and $\mathcal{Z}$ the class with one element of size 1.

6. Translate $(\star)$ into an equation on the ordinary generating function $M(z)$ of Motzkin paths and solve it.

7. Defining $Y(z) := zM(z)$, deduce from the previous question a functional equation for $Y(z)$. Apply Lagrange inversion to determine the number of Motzkin paths of size n.

Note that both classes are enumerated by the same sequence of numbers, called Motzkin numbers.

8. Show a bijective proof of this fact. (Hint: To find the desired bijection, it can be useful to compare the two recursive descriptions of the classes considered, see questions 1 and 5.)

9. Using the "singularity analysis method" find the asymptotic behavior of the Motzkin numbers.