EXERCISE SHEET 8

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Exercise 2

1) $\mathcal{B}$ has no objects of size 0. $B(z)=\sum_{n\geq 1} b_n z^n$

$$\mathcal{A}=\{(\beta,\beta) \mid \beta\in\mathcal{B}\}$$

Each object in $\mathcal{B}$ of size n gives an object of size $2n$ in $\mathcal{A}$

$\Rightarrow$ $b_n = a_{2n}$. Noting also that there are no objects of odd size in $\mathcal{A}$ we conclude that

$$A(z)=\sum_{n\geq 1} a_n z^n = \sum_{n\geq 1} b_n z^{2n} = \sum_{n\geq 1} b_n (z^2)^n = B(z^2).$$

2) $\mathcal{C}=MSet_{\{0,2\}}(\mathcal{B}) \rightsquigarrow C(z)=?$

$$MSet_{\{0,2\}}(\mathcal{B}) = \left\{ \begin{array}{c}\text{Subsets of } \mathcal{B} \text{ containins} \\ 2 \text{ elements}\end{array} \right\} \cup \{\emptyset\}$$

Recalling that $Seq_{\{0,2\}}(\mathcal{B})$ is $1+B(z)^2$ $\left[Seq_{\Omega}(\mathcal{B}) \rightsquigarrow \sum_{k\in\Omega} B(z)^k \right]$ and noting that any non-empty element $\{\alpha,\beta\}$ of $MSet_{\{0,2\}}(\mathcal{B})$ with $\alpha\neq\beta$, corresponds to 2 elements in $Seq_{\{0,2\}}(\mathcal{B})$, i.e. (α,β) and (β,α) we obtain that:

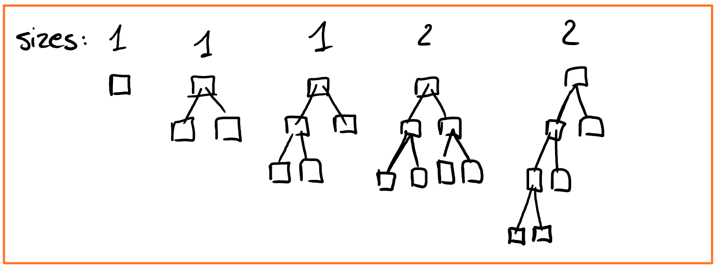
$$C(z) = 1 + \frac{1}{2}\Big(\underbrace{B(z)^2 - B(z^2)}_{\text{g.f. for the elements of } Seq_{\{0,2\}}(\mathcal{B})\setminus\mathcal{A}}\Big) + \underbrace{B(z^2)}_{\text{g.f. for the diagonal}}$$

$$= 1 + \frac{1}{2} B(z)^2 + \frac{1}{2} B(z^2)$$

3) We conclude that

$$U(z) = z\left(1 + \frac{1}{2}U(z)^2 + \frac{1}{2}U(z^2)\right) \quad (*)$$

We can check the equation for the first terms of $U(z)$, indeed noting that $u_1 = u_3 = u_5 = 1$, $u_{2n} = 0$, $u_7 = 2$, we have $U(z) = z + z^3 + z^5 + 2z^7 + O(z^8)$



and substituting in Equation $(*)$, we obtain

$$z + z^3 + z^5 + 2z^7 + O(z^8) = z\left(1 + \frac{1}{2}\underbrace{\left(z + z^3 + z^5 + 2z^7 + O(z^8)\right)^2}_{z^2 + z^6 + 2z^4 + 2z^6 + O(z^8)} - \frac{1}{2}\left(z^2 + z^6 + O(z^8)\right)\right)$$

$$z + z^3 + z^5 + 2z^7 + O(z^8) = z\left(1 + z^2 + z^4 + 2z^6 + O(z^8)\right) \quad \checkmark$$

Exercise 3

We know that, if $C(z)$ denotes the OGF for Dyck paths, then

$$C(z) = 1 + zC(z)^2$$

$\Rightarrow$

$$(C(z) - 1) = z(C(z))^2$$

$\Rightarrow$ Setting $\tilde{C}(z) = C(z) - 1$ and $\phi(u) = (1+u)^2$ we have

$$\tilde{C}(z) = z\,\phi(\tilde{C}(z))$$

Then by the Lagrange inversion formula $(\phi_0 \neq 0)$, we have

$$[z^n]\,\tilde{C}(z) = \frac{1}{n}\,[u^{n-1}]\,\phi(u)^n = \frac{1}{n}\,[u^{n-1}]\,(1+u)^{2n}$$

$$= \frac{1}{n}\,[u^{n-1}]\left(\sum_{k=0}^{2n}\binom{2n}{k}u^k\right) = \frac{1}{n}\binom{2n}{n-1} = \frac{1}{n}\,\frac{(2n)!}{(n-1)!\,(n+1)!}$$

$$= \frac{(2n)!}{n!\,n!}\,\frac{1}{n+1} = \frac{1}{n+1}\binom{2n}{n}$$