EXERCISE SHEET 6

sabato 20 ottobre 2018 10:48

Proof Pringsheim's theorem

Suppose by contradiction that $f(z)$ is analytic at R, i.e., there exists a disc of radius $r>0$ centered at R where f is analytic. We choose h such that $0<h<\frac{r}{3}$ and we consider the expansion of $f(z)$ around $z_0=R-h$

$$(*) \qquad f(z)=\sum_{m\geq 0} g_m (z-z_0)^m$$

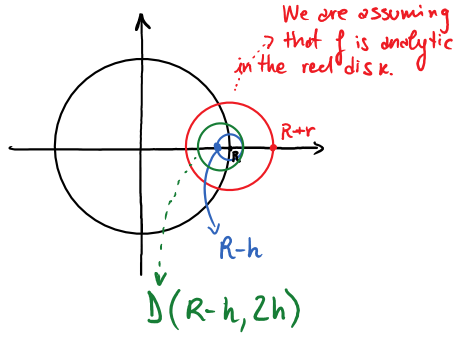
expansion around 0

Since the coefficient of the series expansion around z_0 of an analytic function $f(z)=\sum_{n\geq 0} f_n z^n$ are given by the formula $g_m=\frac{f^{(m)}(z_0)}{m!}$ we have that

$$g_m=\frac{\sum_{n\geq m} f_n \; n\cdot(n-1)\cdots(n-m)\, z_0^{n-m}}{m!}=\sum_{n\geq 0} f_n \binom{n}{m} z_0^{n-m}$$

$\hookrightarrow \binom{n}{m}=0$ if $m>n$

and so $g_m\geq 0$ (since $f_n\geq 0$). Thanks to our choice for h, the series $(*)$ converges at $z=R+h$, therefore



$$f(R+h)=\sum_{m\geq 0}\sum_{n\geq 0} \underbrace{\binom{n}{m} f_n (R-h)^{n-m} (2h)^m}_{\geq 0}$$

$$\overset{!}{=}\sum_{n\geq 0} f_n \left(\sum_{m\geq 0}\binom{n}{m}(R-h)^{n-m}(2h)^m\right)$$

Binomial formula

$$\overset{!}{=}\sum_{n\geq 0} f_n \left[(R-h)+(2h)\right]^n=\sum_{n\geq 0} f_n (R+h)^n$$

This is a contradiction with the fact that the radius of convergence of f is R.

Proof of the "Expansion of meromorphic fonctions":

Around any pole α, $f(z)$ can be expanded locally:

$$f(z)=\sum_{k\geq -M} c_{\alpha,k}(z-\alpha)^k=S_\alpha(z)+H_\alpha(z)$$

"singular part" "analytic part"

$k \geq -M$

"singular part" (all the terms with index in $[-M, \dots, -1]$)

"analytic part"

↳ we can write it as $S_\alpha(z) = \dfrac{N_\alpha(z)}{(z-\alpha)^M}$ with $N_\alpha(z)$ a polynomyal of degree $\leq M$.

Taking the coefficients, we get:

$$[z^n]\, f(z) = \underbrace{[z^n]\big(S(z)\big)}_{\text{we can apply the theorem for rational function}} + [z^n]\big(f(z) - S(z)\big)$$

Therefore it remains to prove that the coefficient $[z^n]\big(f(z)-S(z)\big)$ is $O(R^{-n})$:

analytic for $|z| \leq R$.

analytic at all points of $|z|=R$

$$\Big|[z^n]\big(f(z)-S(z)\big)\Big| \overset{\text{Cauchy's integral}}{=} \frac{1}{2\pi}\left|\int_{|z|=R} \frac{f(z)-S(z)}{z^{n+1}}\, dz\right| \leq \frac{1}{2\pi}\cdot 2\pi R \cdot \sup\left\{\frac{f(z)-S(z)}{z^{n+1}} \,\middle|\, |z|=R\right\}$$

$$\leq \frac{R\cdot C}{R^{n+1}} \qquad \square$$

Proof of the "Asymptotics of coefficients of basic functions":

$(1+z)^\alpha = \sum_{n\geq 0} \binom{\alpha}{n} z^n$

We know that

$$[z^n]\, f(z) = \frac{-\alpha(-\alpha-1)\cdots(-\alpha-n+1)}{n!}\cdot(-1)^n = (-1)^{2n}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\,\Gamma(n+1)} \underset{n\to\infty}{\sim} \frac{n^\alpha\,(n-1)!}{\Gamma(\alpha)\; n!} = \frac{n^{\alpha-1}}{\Gamma(\alpha)}$$

$$f(z) = (1-z)^{-\alpha}$$

$\Gamma(\alpha+n) = (\alpha+n-1)\,\Gamma(\alpha+n-1)$

$\Gamma(\alpha+n) \underset{n\to\infty}{\sim} n^\alpha\, \Gamma(n)$

Proof of the "Transfer theorem" (The Big-Oh case).

The starting point is Cauchy's coefficient formula:

$$f_n = [z^n]\, f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}\, dz$$

where γ is any loop around the origin which is internal to the Δ-domain of f.

We choose $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ (see picture) with $1 < r < R$ and $\phi < \theta < \frac{\pi}{2}$

so that $\mathrm{Supp}(\gamma) \subseteq \Delta(\phi, R)$.

For $j = 1, \dots, 4$ let

$$f_n^{(j)} = \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)}{z^{n+1}}\, dz$$

$\boxed{j=1}$

$$\left| f_n^{(1)} \right| = \frac{1}{2\pi} \left| \int_{\gamma_1} \frac{\overset{O((1-z)^{-\alpha})}{f(z)}}{z^{n+1}}\, dz \right| \underset{\substack{z = 1+w \\ |w| = \frac{1}{n}}}{=} \frac{O\left(\left(\frac{1}{n}\right)^{-\alpha}\right)}{O(1)} \cdot \overbrace{O\left(\frac{1}{n}\right)}^{\text{length of } \gamma_1} = O(n^{\alpha-1})$$

$\boxed{j=2,4}$

$$\left| f_n^{(2)} \right| = \frac{1}{2\pi} \left| \int_{\gamma_2} \frac{f(z)}{z^{n+1}}\, dz \right| \underset{z = 1 + \frac{e^{i\theta} t}{n}}{\leq} \frac{1}{2\pi} \left| \int_1^\infty \frac{f\left(1 + \frac{e^{i\theta} t}{n}\right)}{\left(1 + \frac{e^{i\theta} t}{n}\right)^{n+1}} \frac{e^{i\theta}}{n}\, dt \right| \leq$$

$f(z) \leq K(1-z)^{-\alpha}$ $\qquad dz = \frac{e^{i\theta}}{n} dt$

$$\leq \frac{1}{2\pi} \int_1^\infty K \left| \frac{t}{n} \right|^{-\alpha} \left| 1 + \frac{e^{i\theta} t}{n} \right|^{-n-1} \frac{1}{n}\, dt$$

Noting that

$$\left| 1 + \frac{e^{i\theta} t}{n} \right| \geq 1 + \mathrm{Re}\left(\frac{e^{i\theta} t}{n} \right) = 1 + \frac{t}{n} \cos\theta$$

we have

$$\left| f_n^{(2)} \right| \leq \frac{K}{2\pi} n^{\alpha-1} \underbrace{\int_1^\infty t^{-\alpha} \left(1 + \frac{t}{n}\cos\theta\right)^{-n-1} dt}_{\downarrow\, n \to \infty}$$

$$\int_1^\infty t^{-\alpha} e^{-t\cos\theta}\, dt < \infty \quad \left(\text{since } 0 < \theta < \frac{\pi}{2}\right)$$

Thus $\left| f_n^{(2)} \right| = O(n^{\alpha-1})$

Similarly $\left| f_n^{(4)} \right| = O(n^{\alpha-1})$

Thus $|g_n| = O(n \quad)$

Similarly $|f_n^{(4)}| = O(n^{\alpha-1})$.

$\boxed{j=3}$

$$|f_n^{(3)}| = \frac{1}{2\pi}\left|\int_{\gamma_3}\frac{f(z)}{z^{n+1}}\,dz\right| \le \frac{1}{\cancel{2\pi}}\, C\, \frac{1}{r^{n+1}}\, \cancel{2\pi}\, r = O\left(\left(\frac{1}{r}\right)^n\right) \quad [\text{exponentially small}]$$

$f(z)$ in γ_3 is bounded

We can conclude that $|f_n| = O(n^{\alpha-1})$. $\square$